12th Graph Theory and Algebraic Combinatorics Conference

GTACC12

7-8 February 2024

Department of Mathematics
Faculty of basic Sciences
Tafresh University
Tafresh,Iran

Book of Abstracts

Topics:

Graph Theory
Algebraic Graph Theory
Combinatorics
Algebraic Combinatorics
Algorithmic Graph Theory

Abstracts

12^{th} Graph Theory and Algebraic Combinatorics Conference

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Edited by:

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Preface

Welcome

On behalf of the organizing committee of the 12^{th} Graph Theory and Algebraic Combinatorics Conference (GTACC12), we are pleased to warmly welcome the participants to Tafresh, hoping that their stay will be comfortable and enjoyable.
GTACC12 is held in the department of Mathematics at Tafresh University on 7^{th} and 8^{th} of February 2024. The conference provides a forum for mathematicians, scholar students, and multi-disciplinary researchers to present and discuss their recent results regarding all aspects of Combinatorics, Algebraic Combinatorics, Graph Theory, Algebraic Graph Theory, and Algorithmic Graph Theory.
The secretary office of the conference received 70 submissions for oral presentation where 57 have been accepted by the scientific committee. There are six invited keynote speakers from Iran. We have made every effort to make the present conference as worthwhile as possible.
It is our pleasure to express our thanks to all whose help has made this gathering possible, particularly, Dr. Soheil Vasheghani Farahani the president of Tafresh University for his valuable and unique support and excellent suggestions, the reviewers for their contributions together with providing valuable suggestions and comments to the authors, all the authors of submitted papers, participants, and finally our colleagues in the department of Mathematics, especially, the website manager Dr. Ali Khatibi for his excellent efforts and board of directors for accompaniment and encouragement.
Our special gratitude is going to Mrs. Amene Alidadi, Fateme Eskandari, Soheir Rouhani, and Narges Eshaghi for worthy and significant contributions.

The conference will not achieve its scientific momentum and success without your expertise and active participation.

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Invited Talks

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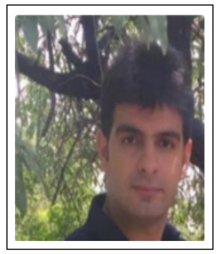

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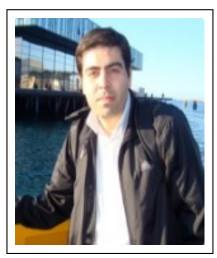

Ali Taherkhani
Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan

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Automorphism group in complex networks

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Abstract

This article examines the symmetry structure of real-world complex networks, a topic that has not been thoroughly explored despite the prevalence of symmetry in such networks. It delves into network symmetry through the automorphism group of the underlying graph and identifies essential symmetries, which are then used to break down the automorphism group into irreducible factors. This breakdown enables efficient handling of large automorphism groups in real-world networks. It associates symmetric subgraphs with each factor in the decomposition and explores their generic structure. Furthermore, by examining automorphism group orbits, it investigates the connection between network symmetry and redundancy. The article also explores the relationship between graph spectra and automorphism groups. The study demonstrates that if a graph has no repeated eigenvalues, then all non-trivial automorphisms are involutions, and its automorphism group is an elementary abelian 2-group. Additionally, if all eigenvalues of a graph are simple, its automorphism group is Abelian. These findings offer valuable insights into the relationship between eigenvalues and automorphisms.

1 Introduction

The use of complex networks to model the underlying topology of "real-world" complex systemsfrom social interaction networks such as scientific collaboration networks [9, 10] to biological regulatory networks [6] and technological networks such as the internet [12]has attracted much current research interest [1], [11], [13]. Previous studies have highlighted the fact that seemingly disparate networks often have certain features in common including (amongst others): the "small-world" property [14]; the power-law distribution of vertex degrees [3]; and network construction from motifs [8].

Identification of universal structural properties such as these allows generic network properties to be decoupled from system-specific features. In this present work we consider the symmetry structure of a variety of real-world networks and find that a certain degree of symmetry is also ubiquitous in complex systems. Although the symmetry structure of some types of well-ordered networks has received some attention [5, 7], a systematic study of the symmetry structure of real-world complex networkswhich typically contain ordered and disordered elementshas not yet been undertaken.

*Speaker

subjclass[2010]: 05C31, 05C60

keywords: Complex network, Symmetry, Automorphism group, graph eigenvalue

This paper therefore investigates the origin and form of real-world network symmetry and its effect on network function. We consider network symmetry via the automorphism group of the underlying graph. Firstly, we identify essential network symmetries and use these symmetries to derive a natural direct product decomposition of the automorphism group into irreducible factors. This decomposition is per se a very efficient way to handle large automorphism groups of real-world networks. We then associate with each factor in this decomposition a symmetric subgraphthe subgraph on which the factor subgroup acts non-triviallyand investigate the generic structure of symmetric subgraphs. Finally, by considering automorphism group orbits we investigate the relationship between network symmetry and redundancy.

As a fundamental object in mathematics and computer science, graphs have various practical applications in different fields such as chemistry [2], biology [17], and economics [4]. One approach to study graphs is through graph invariants which are particular numerical, spatial, or combinatorial properties associated with graphs. Graph invariants are useful in a variety of applications, such as molecular structure prediction, network optimization, and data analysis.

In this article, graphs presented are both simple and connected. This means that they are devoid of any loops or duplicate edges between the same vertices.

Let G be a graph and A its adjacency matrix. The spectrum of G is defined as the set of eigenvalues of A denoted by $Spec(G) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, where n is the order of A. The spectral radius of G is the maximum absolute value of its eigenvalues denoted by λ_1.

The eccenteric version of extended adjacency matrix A, is defined as follows:

$$A_{ex}^{\varepsilon} = \begin{cases} \dfrac{\varepsilon_u^2 + \varepsilon_v^2}{\varepsilon_u \varepsilon_v} & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases}.$$

A vertex-transitive graph is a type of graph in which, for every pair of vertices, there exists an automorphism of the graph that maps one vertex to the other.

2 Main results

In this section, we present some new results on the properties of graphs and their automorphism groups. Our first theorem, which is a well-known result in the literature, states that if a graph has no repeated eigenvalues, then all of its non-trivial automorphisms are involutions, and equivalently, its automorphism group is an elementary abelian 2-group.

Our second theorem establishes a relationship between the automorphism group and the extended adjacency matrix of a graph. Specifically, if $\alpha \in Aut(G)$ is an automorphism of the graph G, then the corresponding permutation matrix P_α satisfies $A_{ex}^{\varepsilon}.P_\alpha = P_\alpha.A_{ex}^{\varepsilon}$.

Our third theorem asserts that if all eigenvalues of G are simple, then the automorphism group $G^* = Aut(G)$ is Abelian. This result follows from our second theorem and the fact that if A_{ex}^{ε} has all distinct eigenvalues, then every non-trivial automorphism has order 2 (i.e., it is an involution).

The two corollaries that follow from our third theorem provide additional insights into the relationship between eigenvalues and automorphisms. The first corollary states that if all eigenvalues of A_{ex}^{ε} are simple, then the automorphism group $Aut(G)$ is Abelian and every non-trivial automorphism has order 2. The second corollary asserts that if η is a simple eigenvalue of A_{ex}^{ε}, then for each automorphism $\sigma \in Aut(G)$, the corresponding permutation matrix P_σ maps an eigenvector corresponding to η to a linear combination of itself and another eigenvector corresponding to η.

Theorem 2.1. *[15, 16, 18] If a multigraph has no repeated eigenvalues then all of its non-trivial automorphisms are involutions; equivalently, its automorphism group is an elementary abelian 2-group.*

Theorem 2.2. *If $\alpha \in Aut(G)$ and P_α is a permutation matrix correspond to α, then $A_{ex}^{\varepsilon}.P_\alpha = P_\alpha.A_{ex}^{\varepsilon}$.*

Theorem 2.3. *Let $G^* = Aut(G)$. If all eigenvalues of G are simple, then G^* is Abelian.*

Corollary 1. Since the matrix A^{ε}_{ex} has all distinct eigenvalues, then the automorphism group $Aut(G)$ is Abelian and every non-trivial automorphism has order 2.

Corollary 2. Suppos η is a simple eigenvalue of A^{ε}_{ex}. Then for each automorphism $\sigma \in Aut(G)$ and permutation matrix P_σ, by Lemma $A^{\varepsilon}_{ex}v = \eta v$ yields that $P_\sigma A^{\varepsilon}_{ex}v = A^{\varepsilon}_{ex}P_\sigma v$. So both v and $P_\sigma v$ are eigenvectors correspond to η. Since $m(\eta) = 1$, we conclude that v and $P_\sigma v$ are linear dependent. Hence, there is an scaler $\mu \in R$

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Investigating a Conjecture Regarding Graph Energy

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Abstract

The energy of a graph G, represented by $\mathcal{E}(G)$, is defined as the total sum of the absolute values of all eigenvalues associated with G. A conjecture was made in (MATCH Commun. Math. Comput. Chem. 83 (2020) 631–633), proposing that for any graph G with a maximum degree $\Delta(G)$ and a minimum degree $\delta(G)$, where the adjacency matrix of G is non-singular, the energy $\mathcal{E}(G)$ is greater than or equal to the sum of $\Delta(G)$ and $\delta(G)$. Furthermore, the conjecture suggests that this equality holds true if and only if G is a complete graph. Researchers have been actively attempting to prove this conjecture for various classes of graphs. Here, we highlight some of their endeavors.

1 Introduction

Consider a graph G with its vertex set denoted as $V(G)$ and its edge set denoted as $E(G)$. The *adjacency matrix* of the graph G of order n, denoted as $A(G) = [a_{ij}]$, is an $n \times n$ matrix. The entry a_{ij} of this matrix is 1 if the vertices v_i and v_j are connected by an edge in $E(G)$, and it is 0, otherwise. The eigenvalues of $A(G)$ are equivalent to the eigenvalues of the graph G. The *spectral radius* of G is defined as the greatest eigenvalue of G. The graph $L(G)$, known as a *line graph*, is derived from the graph G by associating each edge of G with a vertex in $L(G)$. In $L(G)$, two vertices are connected if and only if their corresponding edges in G share a common vertex. A graph that can be depicted on a plane in a manner where the edges only intersect at points representing their shared endpoints is known as a *planar graph*.

The *energy* of the graph G, denoted as $\mathcal{E}(G)$, is defined as the sum of the absolute values of the eigenvalues of $A(G)$. The concept of graph energy was initially introduced by Gutman in 1978 [6]. For further insights into the properties of graph energy, refer to [1, 4, 7]. A; so, various lower bounds exist for the energy of graphs. As instance, Zhou investigated the problem of bounding the graph energy based on the minimum degree and other parameters [9].

2 Main results

In a study conducted by [8], it was demonstrated that for a connected graph G, the energy of G is greater than or equal to twice the minimum degree of G. Furthermore, the equality holds if and only if G is

*Speaker

subjclass[2010]: 05C50, 05C70
keywords: Graph energy, Non-singular graphs, Adjacency matrix

a complete multipartite graph with equal-sized parts. Another improvement on this lower bound was presented in [5], where it was proven that if G is a connected graph with an average degree of $\overline{d}$, then the energy of G is greater than or equal to twice $\overline{d}$. Similarly, the equality holds if and only if G is a complete multipartite graph with equal-sized parts. Additionally in [5], authors put forth an interesting conjecture in the same publication.

Conjecture 2.1. *[5] For any graph G whose adjacency matrix is non-singular, the energy of G is greater than or equal to the sum of the maximum and minimum degrees of G. Also, the equality holds if and only if G is a complete graph.*

According to the research conducted by the authors in a previous study [3], it was demonstrated that the Conjecture 2.1 is valid for planar graphs, graphs without triangles, and graphs without quadrangles.

Theorem 2.2. *[3] The Conjecture 2.1 holds for triangle-free graphs, quadrangle-free graphs and planar graphs.*

Theorem 2.3. *[3] The Conjecture 2.1 is valid for a graph that possesses an integral spectral radius.*

The authors of [2] have made improvements to Conjecture 2.1 regarding line graphs by eliminating the constraint of non-singularity. In the next theorem, they present the enhanced version as follows:

Theorem 2.4. *For any line graph $L(G)$ with a minimum order of 7 and maximum degree $\Delta(L(G))$, along with a minimum degree $\delta(L(G))$, where G is a connected graph,*

$$\mathcal{E}(L(G)) \geq \Delta(L(G)) + \delta(L(G)).$$

The equality condition is satisfied only if $L(G)$ is a complete graph.

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On the chromatic number of almost stable general Kneser hypergraphs

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Abstract

Let $n \geq 1$ and $s \geq 1$ be integers. An almost s-stable subset A of $[n] = \{1, \ldots, n\}$ is a subset such that for any two distinct elements $i, j \in A$, one has $|i - j| \geq s$. For a family $\mathcal{F}$ of non-empty subsets of $[n]$ and an integer $r \geq 2$, the chromatic number of the r-uniform Kneser hypergraph $\mathrm{KG}^r(\mathcal{F})$, whose vertex set is $\mathcal{F}$ and whose edge set is the set of $\{A_1, \ldots, A_r\}$ of pairwise disjoint elements in $\mathcal{F}$, has been studied extensively in the literature and Abyazi Sani and Alishahi were able to give a lower bound for it in terms of the equatable r-colorability defect, $\mathrm{ecd}^r(\mathcal{F})$. In this talk, the methods of Chen for the special family of all k-subsets of $[n]$, are modified to give lower bounds for the chromatic number of almost stable general Kneser hypergraph $\mathrm{KG}^r(\mathcal{F}_s)$ in terms of $\mathrm{ecd}^s(\mathcal{F})$. Here $\mathcal{F}_s$ is the collection of almost s-stable elements of $\mathcal{F}$. We also propose a generalization of a conjecture of Meunier.

1 Introduction and Main results

Let $n \geq 1$, $s \geq 1$ and $r \geq 2$ be integers. Let $\mathcal{F}$ be a family of non-empty subsets of $[n] = \{1, \ldots, n\}$. We say that a subset A of $[n]$ is s-stable if for all distinct elements i and j in A, one has

$$s \leq |i - j| \leq n - s.$$

If we only demand $|i - j| \geq s$, then A is said to be almost s-stable. We use the notation $\mathcal{F}_s$ for the almost s-stable subsets in $\mathcal{F}$. The r-uniform Kneser hypergraph $\mathrm{KG}^r(\mathcal{F}_s)$ is an r-uniform hypergraph whose vertex set is $\mathcal{F}_s$ and whose edge is the set of all pairwise disjoint subsets $\{A_1, \ldots, A_r\}$ in $\mathcal{F}_s$. We use the notion of the equitable r-colorability defect of $\mathcal{F}$, defined by Abyazi Sani and Alishahi in [1]. It is defined as follows.

Definition 1.1. *The r-colorability defect of a family of non-empty subsets $\mathcal{F}$ of $[n]$ is defined to be the minimum size of a subset $X_0 \subseteq [n]$ such that there is an equitable partition*

$$[n] \backslash X_0 = X_1 \cup \cdots \cup X_r$$

so that there are no $F \in \mathcal{F}$ and $1 \leq i \leq r$ such that $F \subseteq X_i$. Here equitable means that $||X_i| - |X_j|| \leq 1$ for all $1 \leq i \leq j \leq r$.

*Speaker

subjclass[2010]:
keywords: Chromatic Number, Tucker's lemma, Topological Graph Theory

Our goal here is to prove the following two theorems.

Theorem 1.2. *If r is a power of 2 and s is a multiple of r, then*

$$\chi(KG^r(\mathcal{F}_s)) \geq \left\lceil \frac{ecd^s(\mathcal{F})}{r-1} \right\rceil.$$

It is plausible to make the following conjecture.

Conjecture 1.3. *For any $n \geq 1$, $r \geq 2$, $s \geq r$ and any family $\mathcal{F}$ of subsets of $[n]$, one has*

$$\chi(KG^r(\mathcal{F}_s)) \geq \left\lceil \frac{ecd^s(\mathcal{F})}{r-1} \right\rceil.$$

Remark 1.4. *This conjecture for the special family of all k-subsets of $\{1, \ldots, n\}$ was made by Meunier in [5]. A version of this conjecture with the topological r-colorability defect was made for a general family and the s-stable part of the family by Frick in [4].*

We also prove the following theorem.

Theorem 1.5. *If $r = p$ is a prime number and $s \geq 2$ is an integer, then*

$$\chi(KG^p(\mathcal{F}_s)) \geq \left\lceil \frac{n - \alpha_1 - \alpha_2}{p-1} \right\rceil$$

where $\alpha_1 = (s-1)\left\lfloor \frac{n - ecd^p(\mathcal{F})}{p} \right\rfloor$ *and* $\alpha_2 = \left\lfloor (p-1)\frac{n - ecd^p(\mathcal{F})+1}{p} \right\rfloor$.

Remark 1.6. *If $p = 2$ and $\mathcal{F}$ is the family of all k-subsets of $[n]$, then $ecd^2(\mathcal{F}) = n - 2(k-1)$, and hence $\alpha_1 = (s-1)(k-1)$ and $\alpha_2 = k-1$. It follows that*

$$\chi(KG^2(\mathcal{F}_s)) \geq n - s(k-1) = ecd^s(\mathcal{F}).$$

This gives a confirmation of the conjecture 1.3 for $r = 2$ and the family of all k-subsets of $[n]$. This was proved by Chen in [3]. Also it is worthwhile to note that if $n \geq sk$, then coloring each almost s-stable k-subset with the value of its minimum element gives a proper coloring of $KG^2(\mathcal{F}_s)$ with $n - s(k-1)$ colors. So, in fact the above inequality is an equality.

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Structural Graph Parameters and Parameterized Algorithms

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Abstract

One approach toward dealing with intractable graph problems is structural parameterization which is defined as investigation of computational complexity of NP-hard graph problems measured as a function of the structural properties of the input graph. The structure of a graph can be measured using some well-known and well-studied parameters such as treewidth, tree-depth, clique-width, vertex cover and neighborhood diversity. The main question, here, is that, for an NP-hard problem, what is the algorithmic cost of generalizing a structural parameter of the input. In this talk, after introducing some of these parameters, we give a short (and not thorough) survey on the tools such as DP and ILP techniques using in investigation of the complexity of hard problems whose input space is confined on a class of graphs with a specified structure (e.g. bounded treewidth graphs). Moreover, we give a brief introduction to the theory of parameterized complexity and classifying NP-hard problems into classes such as FPT and W[1]-hard in terms of their tractability with respect to structural parameters of the input graph.

On the 2-distance injective coloring of graphs

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Abstract

Let $G = \big(V(G), E(G)\big)$ be a graph, and f be a map from $V(G)$ to a set of labels (colors). The map f is said to be a 2-distance coloring of G if the colors of the vertices of any P_3 path are different under f.

An injective coloring of a given graph G is a vertex coloring f of G such that no vertex v is adjacent to two vertices u and w with $f(u) = f(w)$. It has been shown that the injective coloring of G, is not necessarily a proper coloring, and vice versa.

A map $f : V(G) \to \{1, 2, 3, ..., k\}$ is said to be a 2-distance injective k coloring of G if for a vertex v and the vertices u, w, x, y, z, t, there exist a uvw-path of length 2 or a $xyvzt$ path of length 4, then the vertices u, w, x, y, z, t receive distinct colors. In other word, if we observe a path $P_3 = uvw$, or a path $P_5 = xyvzt$ in graph, then $f(u), f(w), f(x), f(y), f(z)$ and $f(t)$ are mutually distinct. The 2-distance injective chromatic number of G, denoted by $\chi_{2i}(G)$, is the least k such that G has a 2-distance injective k-coloring.

In this talk we investigate some properties of 2-distance injective coloring of a graph G. The 2-distance injective coloring versus to 2-distance coloring and injective coloring are investigated. Also 2-distance injective coloring are studied in terms of other parameters such as order, degree, independence number and etc.

1 Introduction

Throughout this paper, we consider G as a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [11] as a reference for terminology and notation which are not explicitly defined here. The *open neighborhood* of a vertex v is denoted by $N_G(v)$, and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. The *minimum* and *maximum degrees* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Finally, for a given set $S \subseteq V(G)$, by $G[S]$ we represent the subgraph of G induced by S. For any two vertices u and v of G, we denote $d_G(u, v)$ the distance between u and v, that is the length of a shortest path joining u and v.

Graph coloring has many applications in various fields of life, such as timetabling, scheduling daily life activities, scheduling computer processes, registering allocations to different institutions and libraries,

*Speaker

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keywords: injective coloring, 2-distance coloring, 2-distance injective coloring

manufacturing tools, printed circuit testing, routing and wavelength, bag rationalization for a food manufacturer, satellite range scheduling, and frequency assignment. These are some applications out of the many that already exist and many to come. In fact, coloring has inspired many other fields.

A k-coloring of the vertices of $G = (V, E)$ a graph is a function $f : V(G) \to \{1, 2, 3, ..., K\}$. A k-coloring f is a proper coloring, if and only if, for all edge $xy \in E$, $f(x) \neq f(y)$. In other words, the colors of the vertices of all P_2 paths in the graph are distinct. The chromatic number of G, denoted by $\chi(G)$, is the minimum integer k such that G has a proper k-coloring.

The study of distance coloring was initiated by Kramer and Kramer ([7] and [6]) in 1969. A 2-*distance coloring* (or, 2DC for short) of a graph G is a mapping of $V(G)$ to a set of colors (nonnegative integers for convenience) in such a way that any two vertices of distance at most two have different colors. The minimum number of colors (nonnegative integers) k for which there is a 2DC is called the 2-*distance chromatic number* $\chi_2(G)$ of G. For any 2DC $f : V(G) \to \{1, \cdots, k\}$, we write $f = (V_1^f, \cdots, V_k^f)$ in which $V_i^f = \{v \in V(G) : f(v) = i\}$ for each $1 \leq i \leq k$. If there is no confuse with respect to the mapping f, we omit the superscripts f and only write $f = (V_1, \cdots, V_k)$.

The injective coloring was first introduced in 2002 by Hahn, Kratochvil, Siran, and Sotteau. An injective k-coloring of a graph $G = (V, E)$ is a function $f : V(G) \to \{1, 2, 3, ..., k\}$ such that no vertex v is adjacent to two vertices u and w with $f(u) = f(w)$. In other words, we say that a colouring f of a graph is injective if its restriction to the neighbourhood of any vertex is injective. Also we can say, if xyz is a P_3 path in graph G, then $f(x) \neq f(z)$. The injective chromatic number of G, denoted by $\chi_i(G)$, is the minimum integer k such that G has an injective k-coloring. The injective coloring of G, is not necessarily proper coloring. There are several references on injective coloring, [1, 2, 3, 4, 5, 8, 9, 10].

An obvious alternate way of looking at the injective chromatic number of a graph G is to consider the common neighbour graph $G^{(2)}$ of G defined by $V(G^{(2)}) = V(G)$ and

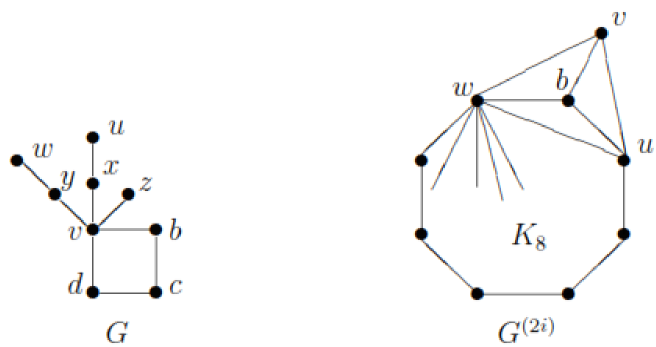
$$E(G^{(2)}) = \{uv : \textit{there is a path of length 2 in G joining u and v}\}$$

. Then, $\chi_i(G) = \chi(G^{(2)})$.

Let u and w be two vertices and we say that u is a 2-distance neighborhood of w if $d_G(u, w) = 2$. We say that two vertices x, y have a common vertex v in a 2-distance neighborhood if there is a path $P_5 = xuvwy$ in graph.

Definition 1.1. *A map $f : V(G) \to \{1, 2, 3, ..., k\}$ is said to be a 2-distance injective k coloring of G if for a vertex v and the vertices u, w, x, y, z, t, there exist a uvw-path of length 2 or a $xyvzt$ path of length 4, then the vertices u, w, x, y, z, t receive distinct colors. In other word, if we observe a path $P_3 = uvw$, or a path $P_5 = xyvzt$ in graph, then $f(u), f(w), f(x), f(y), f(z)$ and $f(t)$ are mutually distinct.*
The 2-distance injective chromatic number of G, denoted by $\chi_{2i}(G)$, is the least k such that G has a 2-distance injective k-coloring. The 2-distance injective chromatic number of G, denoted by $\chi_{2i}(G)$, is the least k such that G has a 2-distance injective k-coloring. Also the 2-distance injective coloring of G, is not necessarily proper coloring, and vice versa.

An alternate way of looking at the 2-distance injective chromatic number of a graph G is to consider a graph $G^{(2i)} = (V^{(2i)}(G), E^{(2i)}(G))$, defined by vertex set, $V^{(2i)}(G) = V(G)$ and edge set as: for $u, w \in V^{(2i)}(G)$, $uw \in E^{(2i)}(G) = E(G^{(2i)})$ if and only if there exists a path uvw, or a path $uxvyw$, or two paths uvz and $ptvrw$ in graph.
See the following example.



It can be easy to see that, if $f = \chi_{2i}$-chromatic map of G, then $f(v) = 1 = f(b)$, $f(c) = 2$, $f(d) = 3$, $f(z) = 4$, $f(u) = 5$, $f(x) = 6$, $f(y) = 7$, $f(w) = 8$.
Also we can see $\chi(G^{(2i)}) = 8$.

Let v be a vertex of G. Then we define $N^{(2)}(v)$ as the set of vertices of at most 2-distance neighborhood of v and $|N^{(2)}(v)|$ as (*1-2*)-distance degree of v in G, $deg_G^{(1\text{-}2)}(v) = deg^{(1\text{-}2)}(v)$. Let $\delta^{(1\text{-}2)}(G)$ and $\Delta^{(1\text{-}2)}(G)$ denote the minimum and maximum (*1-2*)-distance degree of G respectively. It is clear $\Delta(G) \le \Delta^{(1\text{-}2)}(G)$.

2 Main results

Let G be a graph. Then $\chi(G) \le \chi_{2i}(G)$.

Proposition 2.1. *Let G be a graph of diameter at least 4. Then $\chi_2(G) \le \chi_{2i}(G)$. For complete graph K_n, ($n \ne 2$ and cycle C_n, ($n \ne 4, 10$) we have $\chi_2(G) = \chi_{2i}(G)$.*

One of the problem may be as follows: Characterize graph G, with $\chi_2(G) = \chi_{2i}(G)$.

A k-independent set of a graph G, is a subset, $\mathcal{I}_k$, of the vertices of G such that the distance between any two vertices of $\mathcal{I}_k$ in G is at least $k+1$. Let $\mathcal{I}_2$ be the 2-independent set of G and α_2 be the size of the maximum 2-independent set of G.

Proposition 2.2. *Let G be a graph of diameter 4 with independence number α_2. Then, $\chi_{2i}(G) \ge \alpha_2$.*

Proposition 2.3. *Let G be a graph of diameter at least 3 Then, $\chi_{2i}(G) < |V(G)|$.*

Proposition 2.4. *Let G be a graph. Then, $\chi_{2i}(G) = |V(G)|$ if and only if diameter of G is at most 4 and every edge of G is in a C_5.*

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G-free coloring of graphs: a Catlin-type result and a generalization of the Borodin-Kostochka Conjecture

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Abstract

Let G be a connected graph of order at least 2. A G-free k-coloring of a graph H is a partition of the vertex set of H into $V_1, \dots, V_k$ such that $H[V_i]$, the subgraph induced on V_i, does not contain any subgraph isomorphic to G. The G-free chromatic number of H, denoted by $\chi_G(H)$, is the minimum number k for which H has a G-free k-coloring. As an extension of Brooks' Theorem, in 1979, Catlin showed that if H is neither an odd cycle nor a complete graph, then H has a proper $\Delta(H)$-coloring for which one of the color classes is a maximum independent set of H. In this talk, we show that a Catlin-type theorem holds for G-free k-coloring of graphs.

In 1977, Borodin and Kostochka conjectured that any graph H with maximum degree $\Delta(H) \geq 9$ and without $K_{\Delta(H)}$ as a subgraph has chromatic number at most $\Delta(H) - 1$. As an extension of this conjecture, we pose the following natural question.

Question. Suppose that H is a graph with maximum degree $\Delta(H)$ and clique number $\omega(H)$ such that $\omega(H) \leq \Delta(H) - 1$. Assume that $p_1 \geq p_2 \geq \cdots \geq p_k \geq 2$ are k positive integers and $\sum_{i=1}^{k} p_i = \Delta(H) - 1 + k$. Is there a partition of $V(H)$ into $V_1, V_2, \dots, V_k$ such that for each $1 \leq i \leq k$, $H[V_i]$ is K_{p_i}-free?

We provide a positive answer to this question when $p_1 + p_2 \geq 7$. This result directly implies $\chi_{K_p}(H) \leq \lceil \frac{\Delta(H)-1}{p-1} \rceil$ for $p \geq 4$ (where K_p is a complete graph on p vertices). When $p_1 + p_2 \leq 6$, we offer negative answers for some of cases.

This is a joint work with Yaser Rowshan.

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*Speaker

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keywords: Graph coloring, G-free coloring, Brooks' Theorem, The Borodin-Kostochka Conjecture

Contributed Talks

Speaker	Time	Title	Hall
M. Afkhami	Thursday, 8 Feb 2024- 9:30-10:10	A generalization of the idempotent-divisor graphs	D
M. Afkhami	Wednesday, 7 Feb 2024- 11:20-11:40	The line graph co-maximal graphs	B
M. Ahmadi	Thursday, 8 Feb 2024- 11:10-11:30	NP-Completeness of k-total limited packing in graphs	A
F. Rahmani	Wednesday, 7 Feb 2024- 11:40-12	A note on the lower bounds for the energy of graphs	D
F. Aghaei	Wednesday, 7 Feb 2024- 12-12:20	Some results on domination polynomial of neighberhood corona of two graphs	A
A. Alidadi	Thursday, 8 Feb 2024- 11:10-11:30	Introducing a novel vertex-degree-based topological index	E
A. Alidadi	Thursday, 8 Feb 2024- 10:30-10:50	Dominating sets and Sombor index of unicyclic graphs	E
Z. Mohammadpoor	Thursday, 8 Feb 2024- 10:50-11:10	Extremal transmission regular and transmission graph with respect to Wiener index	D
A. Tadayyonfar	Wednesday, 7 Feb 2024- 16:30-16:50	Symmetric and Hemi-Cartesian product with application in zero divisor graph	B
M. Arezoomand	Wednesday, 7 Feb 2024- 11-11:20	Recent results on perfect state transfer problem on Cayley graph	B
M. Aryan	Wednesday, 7 Feb 2024- 12-12:20	Generalized splitting and element splitting operations on p -matroids	C
M. Ghasemi	Wednesday, 7 Feb 2024- 15:50-16:10	The g-good neighbor diagnosability of triangle-free graphs	C
M. Bahrami	Wednesday, 7 Feb 2024- 15:50-16:10	An novel method for computing dot product dimension of certain disconnected graphs with efficient search	D
A. Jafarzadeh	Wednesday, 7 Feb 2024- 11:20-11:40	Mixed partitions of a set and Stirling numbers of the second kind	C
M. Barzegar	Wednesday, 7 Feb 2024- 12:00-12:20	On the regularity of some binomial ideals associated to some specific graphs	B
S. Seifi	Wednesday, 7 Feb 2024- 11-11:20	Saturation number of fullerene graphs	E
M. Bozorgzadeh	Thursday, 8 Feb 2024- 10:10-10:30	Nil clean graphs over Zp rings	D
M. Darougheh	Wednesday, 7 Feb 2024- 16:10-16:30	Laplacian eigenvalues distribution and graph parameters	A
K. Dastouri	Wednesday, 7 Feb 2024- 11:40-12:00	Representation theory of symmetric groups and some combinatorial results	B

Speaker	Time	Title	Hall
M.J. Nadjafi Arani	Thursday, 8 Feb 2024- 09:50-10:10	Organized Fraud detection using Graph Theory and Poisson Process	E
A. Nemayande	Wednesday, 7 Feb 2024- 11:20-11:40	Complete forcing number of tori and hypercube	E
M. Oghbaei	Wednesday, 7 Feb 2024- 11:20-11:40	On the Scale-embedding of weighted graphs into hypercubes	E
M. Nouri Jouybari	Thursday, 8 Feb 2024- 10:10-10:30	Degree distance and Gutman index of graph	E
M. Farkhondeh	Wednesday, 7 Feb 2024- 11:20-11:40	On characteristic and Laplacian characteristic polynomials of some caterpillars	A
A. Fazlikhani	Wednesday, 7 Feb 2024- 11:00-11:20	Characterization of graphs based on chromatic symmetric function	A
M. Ghanbari	Thursday, 8 Feb 2024- 10:50-11:10	A generalization of 3-rainbow domination in graphs	C
V. Ghorbani	Wednesday, 7 Feb 2024- 16:10-16:30	A Conjecture about matroid spikes	C
K. Hamidizadeh	Wednesday, 7 Feb 2024- 15:50-16:10	The gyr-commuting graph of gyrogroups	B
A. Jafari	Wednesday, 7 Feb 2024- 16:30-16:50	k-coalition in graphs	A
S. S. Karimizad	Thursday, 8 Feb 2024- 10:10-10:30	On diameters and domination number in graphs	C
M. Korivand	Thursday, 8 Feb 2024- 10:10-10:30	Breaking symmetry in Graphs by Resolving Sets	B
F. Mirzaei	Wednesday, 7 Feb 2024- 11:40-12:00	The relation between perfect star packing and efficient dominating set in fullerene graph	A
M. Mirzargar	Thursday, 8 Feb 2024- 09:50-10:10	Groups with the same enhanced power graphs	B
M. Mohammadi	Wednesday, 7 Feb 2024- 12:00-12:20	Properties of Sombor Index Matrix of Some Graphs	E
M. Mohammadi	Wednesday, 7 Feb 2024- 16:30-16:50	Sombor index of Some Cactus Chain Graphs	E
N. Nemati	Thursday, 8 Feb 2024- 10:10-10:30	Independent domination gap of a graphs	A
H. Nouri Samani	Thursday, 8 Feb 2024- 10:30-10:50	Global dominator coloring of graphs	A
S. Rabizadeh	Thursday, 8 Feb 2024- 10:30-10:50	Some results on the Sombor index and energy of graphs	D
G. Raeisi	Wednesday, 7 Feb 2024- 11:00-11:20	On the Ramsey number of stars versus an odd wheel	C
A. Rezaei	Wednesday, 7 Feb 2024- 16:10-16:30	The generalization of Cayley graph	D
N. Rezaei	Wednesday, 7 Feb 2024- 16:30-16:50	Prefect 1-factorisations of complete k-uniform hypergraphs	D
S. Rouhani	Wednesday, 7 Feb 2024- 16:10-16:30	Some bounds for the energy of graph and Sombor index	E
Y. Rowshan	Thursday, 8 Feb 2024- 09:50-10:10	The m barpartite Ramsey number $BR_m(K_{2,2}, K_{5,5})$	C

Speaker	Time	Title	Hall
Y. Rowshan	Wednesday, 7 Feb 2024- 15:50-16:10	Some results in list $\mathcal{G}$-free colouring of graph	A
Y. Sadatrasul	Thursday, 8 Feb 2024- 10:50-11:10	The annihilator graph of the Power Set Ring of a Set	B
B. Salehian Matykolaei	Wednesday, 7 Feb 2024- 12:00-12:20	D-Graphs	D
B. Samimi	Thursday, 8 Feb 2024- 09:50-10:10	2-Coupon Coloring of Cubic Graphs Containing 3-Cycle or 4-Cycle	C
H. Saveh	Wednesday, 7 Feb 2024- 11:00-11:20	Some Notes on the Energy of Graphs with Self-Loops	D
H. Soroush	Wednesday, 7 Feb 2024- 16:10-16:30	Degree saving group of simple graph and its characterization	B
A. Taheri	Thursday, 8 Feb 2024- 10:50-11:10	A new method for computation of Frobenius number	A
M. Taheri	Wednesday, 7 Feb 2024- 15:50-16:10	Matching in (3,6)-fullerene graphs	E
E. Tohidi	Wednesday, 7 Feb 2024- 11:40-12:00	Lower bounds for the Randic index of graphs in Terms of Matching Number	E
M. Valinavaz	Thursday, 8 Feb 2024- 10:30-10:50	The $\mathbb{N}_k$-valued Roman domination numbers of graphs	D
Z. Zare	Thursday, 8 Feb 2024- 11:10-11:30	Connected edge cover polynomial of a graph	B
M. Ziaee	Wednesday, 7 Feb 2024- 11:40-12	On the distance transitivity of the bipartite Kneser graphs	C

A generalization of the idempotent-divisor graphs

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Abstract

In this paper, we introduce a generalization of the idempotent-divisor graph of a commutative ring R by using the lower triangular matrices. Assume that R is a commutative ring with nonzero identity. For $n \in \mathbb{N}$, we denote the set of all lower triangular matrices of order n with entries in R by $\mathcal{M}_n^{LT}(R)$. We consider $\mathcal{A}$ to be a subset of $\mathcal{M}_n^{LT}(R)$ consists of all lower triangular matrices whose entries on the main diagonal are non-zero. Let e be an idempotent element of R. For $n \in \mathbb{N}$, we define the graph $\Gamma_e^n(R)$ as a simple graph with vertex set

$$\{X \in R^n \mid \exists Y \in R^n, \exists A \in \mathcal{A}; XAY^T = e \text{ or } YAX^T = e\}$$

and two distinct vertices X and Y are adjacent whenever there exists $A \in \mathcal{A}$ such that $XAY^T = e$ or $YAX^T = e$. In this paper, we study the structure of the graph $\Gamma_e^n(R)$, which is a a generalization of the idempotent-divisor graph.

1 Introduction

Let R be a commutative ring with nonzero identity. By $Z(R)$, $U(R)$ and $E(R)$, we mean the set of all zero-divisor, unit and idempotent elements of R, respectively. The concept of a zero-divisor graph of a commutative ring was introduced by Beck in [3]. However, he let all elements of a ring R be the vertices of the graph and was mainly interested in colorings. In [2], Anderson and Livingston introduced and studied the zero-divisor graph, which is denoted by $\Gamma(R)$, as a simple graph with vertex set $Z(R) \setminus \{0\}$ and two distinct vertices a and b are adjacent if and only if $ab = 0$. Assume that $e \in E(R)$. In [4], Kimball and LaGrange defined the concept of an idempotent-divisor graph $\Gamma_e(R)$ of R associated with e as a generalization of the zero-divisor graph. $\Gamma_e(R)$ is a simple graph with vertex-set $\{a \in R \mid \exists b \in R;\ ab = e\}$ and two distinct vertices a and b are adjacent if and only if $ab = e$. Note that $\Gamma_e(R)$ is the zero-divisor graph of R when $e = 0$.

Let $n \in \mathbb{N}$. We use the notation $\mathcal{M}_n^{LT}(R)$ for the set of all lower triangular matrices of order n whose entries are the elements of R. Let $\mathcal{A}$ be the subset of $\mathcal{M}_n^{LT}(R)$ consists of all lower triangular matrices of order n whose entries on the main diagonal are non-zero. Assume that $e \in E(R)$ is an idempotent of R. For $n \in \mathbb{N}$, we define the graph $\Gamma_e^n(R)$ as a simple graph where $X \in R^n$ is a vertex of $\Gamma_e^n(R)$ if there exists

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keywords: Idempotent-divisor graph, Lower triangular matrix, Commutative ring

$A \in \mathcal{A}$ and $Y \in R^n$ such that $XAY^T = e$ or $YAX^T = e$. Also, two distinct vertices X and Y are adjacent in $\Gamma_e^n(R)$ if and only if there exists $A \in \mathcal{A}$ such that $XAY^T = e$ or $YAX^T = e$. Note that if $n = 1$, then $\Gamma_e^1(R)$ is isomorphic to $\Gamma_e(R)$, and so $\Gamma_e^n(R)$ is a generalization of the idempotent-divisor graph. In [1], the authors provide a generalization of the zero divisor graphs by using matrix theory, and if we ignore the isolated vertices of this graph, then it is isomorphic to $\Gamma_0^n(R)$. Hence $\Gamma_e^n(R)$ is also a generalization of the graph which is introduced in [1]. So in this paper, we assume that $n \geqslant 2$ and $e \neq 0$, and we study the properties of $\Gamma_e^n(R)$.

We use the standard terminology of graphs [5]. Let G be a graph and $V(G)$ be the set of vertices of G. For two distinct vertices x and y, let $d(x,y)$ denote their distance, that is, the length of the shortest path between x and y (we set $d(x,y) := \infty$ if there is no such path). The diameter of G is $\mathrm{diam}(G) = \sup\{d(x,y);\ x \text{ and } y \text{ are distinct vertices of G}\}$. The girth of G is the length of the shortest cycle in G, denoted by $\mathrm{gr}(G)$ ($\mathrm{gr}(G) = \infty$ if G has no cycles). Also, a graph G is called planar if it can be drawn in the plane without any edge crossing. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. A clique of a graph is a complete subgraph of it and the number of vertices in a largest clique of G is called the clique number of G and is denoted by $\omega(G)$.

2 Main results

In this section, we investigate some properties of the graph $\Gamma_e^n(R)$, when R is a finite ring. We begin with the following lemma.

Lemma 2.1. *Assume that $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ are two distinct vertices of $\Gamma_e^n(R)$ such that there exists $1 \leqslant k \leqslant n$ with $x_k, y_k \in U(R)$. Then X and Y are adjacent in the graph $\Gamma_e^n(R)$.*

Notation 2.2. *Let $1 \leqslant i \leqslant n$ and X be a nonempty subset of R. We set*

$$X^{(i)} := \{(a_1, \ldots, a_i, \ldots, a_n) \in R^n; a_i \in X\}.$$

Corollary 2.3. *The following ststements hold.*

(i) For each $1 \leqslant i \leqslant n$, the induced subgraph of $\Gamma_e^n(R)$ with vertex set $U(R)^{(i)}$, forms a clique in the graph $\Gamma_e^n(R)$.

(ii) $\omega(\Gamma_e^n(R)) \geqslant |U(R)||R|^{n-1}$.

Theorem 2.4. *In the graph $\Gamma_e^n(R)$, we have diam$(\Gamma_e^n(R)) \geqslant 2$ or diam$(\Gamma_e^n(R)) = \infty$.*

Theorem 2.5. *Let R be a finite ring. Then gr$(\Gamma_e^n(R)) = 3$ if and only if $n \geqslant 3$ or $|R| \geqslant 3$. Otherwise, gr$(\Gamma_e^n(R)) = \infty$.*

Theorem 2.6. *If the graph $\Gamma_e^n(R)$ is planar, then we have $|U(R)| = 1$ and $n \leqslant 3$.*

Proposition 2.7. *If R is a finite local ring, then $\Gamma_e^n(R)$ is isomorphic to $\Gamma_1^n(R)$.*

Let $\mathbb{Z}_n = \{\overline{1}, \overline{2}, \ldots, \overline{n-1}\}$ be the ring of integers module $n \geqslant 2$. In the rest of this section, we determine the vertices of the graph $\Gamma_{\overline{1}}^2(\mathbb{Z}_n)$. For two integer s and t, the greatest common divisor of s and t is denoted by $gcd(s,t)$. Recall that for a non-zero element $\overline{x}$ of $\mathbb{Z}_n$, $\overline{x}$ is a unit element if and only if $gcd(x,n) = 1$ and also every non-unit element of $\mathbb{Z}_n$, is a zero divisor. So, the number of zero divisor elements of $\mathbb{Z}_n$ is equal to $n - \phi(n)$, where ϕ is the Euler function. We say d is a proper divisor of n, when $1 < d < n$ and $d|n$. Let d_1, d_2, …, d_k be distinct proper divisor of n. For $1 \leqslant i \leqslant k$, we define $A_{d_i} = \{\overline{x} \in \mathbb{Z}_n \mid gcd(x,n) = d_i\}$. It is easy to see that A_{d_1}, A_{d_2}, …, A_{d_k} are pairwise disjoint and

$$Z(\mathbb{Z}_n) \setminus \{0\} = A_{d_1} \cup A_{d_2} \cup \ldots \cup A_{d_k}.$$

Lemma 2.8. [6, Proposition 1.2] *For $1 \leqslant i \leqslant k$, $|A_{d_i}| = \phi(\frac{n}{d_i})$.*

Lemma 2.9. *The element* $(\overline{x}, \overline{y}) \in \mathbb{Z}_n^2$ *is a vertex of the graph* $\Gamma_{\overline{1}}^2(\mathbb{Z}_n)$ *if and only if* $\overline{gcd(x,y)} \in U(\mathbb{Z}_n)$.

Proposition 2.10. *The following statements hold.*

(a) *For* $(\overline{x_1}, \overline{x_2}) \in \mathbb{Z}_n^2$, *if* $\overline{x_1} \in U(\mathbb{Z}_n)$ *or* $\overline{x_2} \in U(\mathbb{Z}_n)$, *then* $(\overline{x_1}, \overline{x_2})$ *is a vertex of the graph* $\Gamma_{\overline{1}}^2(\mathbb{Z}_n)$.

(b) *For* $(\overline{x_1}, \overline{x_2}) \in A_{d_i}^2$, *where* $1 \leqslant i \leqslant k$, $(\overline{x_1}, \overline{x_2})$ *is not a vertex of the graph* $\Gamma_{\overline{1}}^2(\mathbb{Z}_n)$.

(c) *For* $(\overline{x_1}, \overline{x_2}) \in A_{d_i} \times A_{d_j}$, *where* $1 \leqslant i \neq j \leqslant k$, $(\overline{x_1}, \overline{x_2})$ *is a vertex of the graph* $\Gamma_{\overline{1}}^2(\mathbb{Z}_n)$ *if and only if* $gcd(d_i, d_j) = 1$.

(d) *The elements* $(\overline{x_1}, \overline{0})$ *and* $(\overline{0}, \overline{x_2})$ *are the vertices of the graph* $\Gamma_{\overline{1}}^2(\mathbb{Z}_n)$ *if and only if* $\overline{x_1}, \overline{x_2} \in U(\mathbb{Z}_n)$.

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On the line co-maximal graphs

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Abstract

Let R be a commutative ring with nonzero identity. The co-maximal graph of R, denoted by $\Gamma(R)$, is a simple graph with all elements of R as vertices and two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. Let $\Gamma_1(R)$ and $\Gamma_2(R)$ be the induced subgraphs of $\Gamma(R)$ with vertex sets $U(R)$ and $R \setminus U(R)$, respectively. In this paper, we completely characterize when the graphs $\Gamma(R)$, $\Gamma_1(R)$, $\Gamma_2(R)$ and $\Gamma_2(R) \setminus J(R)$ are line graphs or complement of line graphs.

1 Introduction

Let R be a commutative ring with non-zero identity. Let $U(R)$, $\mathrm{Max}(R)$ and $J(R)$ be the set of all unit elements, the set of maximal ideals of R and the Jacobson radical of R, respectively. In [3], Sharma and Bhatwadekar defined the co-maximal graph of R, denoted by $\Gamma(R)$, which is a simple graph with all elements of R as vertices and two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. Let $\Gamma_1(R)$ and $\Gamma_2(R)$ be the induced subgraphs of $\Gamma(R)$ with vertex sets $U(R)$ and $R \setminus U(R)$, respectively. Clearly $\Gamma_1(R)$ is a complete subgraph of $\Gamma(R)$. In [2], the authors studied the subgraph $\Gamma_2(R)$ and also they investigate some properties of the graph $\Gamma_2(R) \setminus J(R)$, which is denoted by $\Gamma_3(R)$.

We use the standard terminology of graphs [4]. Let G be a graph. The line graph $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident in G. Also, the complement of G, denoted by $\overline{G}$, is the graph has the same vertex set as G but whose edge set consists of the edges not present in G. In this paper, we investigate when the graphs $\Gamma(R)$, $\Gamma_1(R)$, $\Gamma_2(R)$ and $\Gamma_3(R)$ are line graphs. Also, we study when the co-maximal graph is the complement of a line graph.

Throughout the paper, R is a finite commutative ring with non-zero identity and $\mathbb{F}$ is a finite field. The field with q elements is denoted by $\mathbb{F}_q$. Also, $\mathbb{Z}_n$ denotes the ring of integers modulo n.

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*Speaker

subjclass[2010]: 05C75, 13A15
keywords: Co-maximal graph, Line graph, Commutative ring

2 Main results

In this section, we study when the graphs $\Gamma(R)$, $\Gamma_1(R)$, $\Gamma_2(R)$ and $\Gamma_3(R)$ are line graphs or complement of line graphs. In fact, we want to classify all finite commutative rings, whose co-maximal graphs are line graphs or complement of line graphs. In order to do this, we will use one of the characterizations of line graphs which was proved in [1].

Theorem 2.1. *[1] Let G be a graph. Then G is the line graph of some graph if and only if none of the nine graphs in Figure 1 is an induced subgraph of G.*

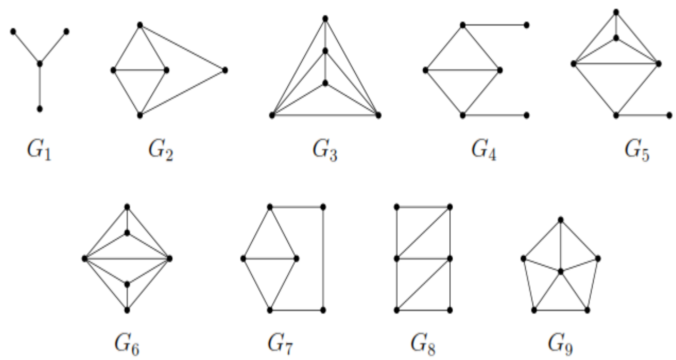

Figure 1:

Theorem 2.2. *Let R be a commutative ring with non-zero identity. Then $\Gamma_1(R)$ is a line graph and also $\Gamma(R)$ is a line graph if and only if R is one of the following rings.*

$$\mathbb{F},\ \mathbb{Z}_4,\ \mathbb{Z}_2[X]/(x^2),\ \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Theorem 2.3. *Let $R \cong R_1 \times R_2 \times \ldots \times R_n$. Then $\Gamma_2(R)$ and $\Gamma_3(R)$ are line graphs if and only if one of the following statements holds.*

(i) R is one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $Z_2 \times \mathbb{Z}_4$ or $Z_2 \times (\mathbb{Z}_2[x]/(x^2))$.

(ii) $n = 1$.

In the rest of this section, we investigate when the graphs $\Gamma(R)$, $\Gamma_1(R)$, $\Gamma_2(R)$ and $\Gamma_3(R)$ are the complement of a line graph. To do this, we use the following version of Theorem 2.1.

Theorem 2.4. *[1] A graph G is the complement of a line graph if and only if none of the nine graphs $\overline{G_i}$, $i = 1, 2, \ldots, 9$, of Figure 2 is an induced subgraph of G.*

Theorem 2.5. *Let R be a commutative ring with non-zero identity. Then the co-maximal graph $\Gamma(R)$ and its subgraph $\Gamma_2(R)$ are the complement of a line graph if and only if one of the following statements holds.*

(i) $|Max(R)| = 2$ and $J(R) = \{0\}$.

(ii) R is a local ring.

Proposition 2.6. $\Gamma_1(R)$ *is the complement of a line graph.*

Theorem 2.7. *Let $R \cong R_1 \times R_2 \times \ldots \times R_n$. Then $\Gamma_3(R)$ is the complement of a line graph if and only if one of the following statements holds.*

(i) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(ii) $n = 1$ or $n = 2$.

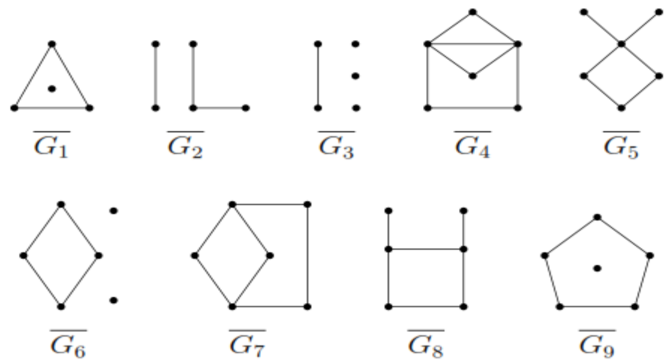

Figure 2:

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NP-completeness of k-total limited packings in graphs

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Abstract

Let $G = \big(V(G), E(G)\big)$ be a graph. A set $B \subseteq V(G)$ is said to be a *k-total limited packing* in the graph G if $|B \cap N(v)| \leq k$ for each vertex v of G. The *k-total limited packing number* $L_{k,t}(G)$ is the maximum cardinality of a k-total limited packing in G.

Here we show that The k-TOTAL LIMITED PACKING problem is NP-complete even for bipartite graphs and for chordal graphs.

1 Introduction

Throughout this manuscript, we consider G as a finite simple graph with a vertex set $V(G)$ and edge set $E(G)$. The *order* of graph is denoted by n and the *size* of graph is m.

The *open neighborhood* of a vertex v is denoted by $N(v)$, and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *minimum* and *maximum degrees* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The subgraph induced by $S \subset V(G)$ in a graph G is denoted by $G[S]$.

A set $S \subseteq V$ is a dominating set if every vertex in $V \backslash S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G.

A set of vertices $B \subseteq V(G)$ is called a *packing* (resp. an *open packing*) in G provided that $N[u] \cap N[v] = \varnothing$ (resp. $N(u) \cap N(v) = \varnothing$) for each distinct vertices $u, v \in V(G)$. The maximum cardinality of a packing (resp. open packing) is called the *packing number* (resp. *open packing number*), denoted $\rho(G)$ (resp. $\rho_o(G)$). For more information about these topics, the reader can consult [2] and [3]. In 2010, Gallant et al. ([1]) introduced the concept of limited packing in graphs. In fact, a set $B \subseteq V(G)$ is said to be a *k-limited packing*in the graph G if $|B \cap N[v]| \leq k$ for each vertex v of G. The *k-limited packing number* $L_k(G)$ is the maximum cardinality of a kLP in G. They also exhibited some real-world applications of it in network security, market situation, NIMBY and codes. This concept was next investigated in many papers. Similarly, a set $B \subseteq V(G)$ is said to be a *k-total limited packing* if $|B \cap N(v)| \leq k$ for each vertex v of G. The *k-total limited packing number* $L_{k,t}(G)$ is the maximum cardinality of a kTLP in G. This concept was first studied in [5]. It is easy to see that the latter two concepts are the same with the concepts of packing and open packing when $k = 1$.

The *corona product* $G \odot H$ of two graphs G and H is obtained by taking one copy of G and $|V(G)|$ copies of H, and joining $v_i \in V(G)$ to every vertex in the ith copy of H.

*Speaker

subjclass[2010]: 05C69, 05C99

keywords: k-total limited packing set, NP-hard, NP-complete

Here we prove the problem of computing the k-total limited packing number is NP-hard, even for some special families of graphs.

2 Main result

We consider the problem of deciding whether a graph G has a k-total limited packing set of cardinality at least a given integer. That is stated in the following decision problem.

k-TOTAL LIMITED PACKING problem
INSTANCE: A graph $G = \big(V(G), E(G)\big)$ and a positive integer z'.
QUESTION: Does G have a k-total limited packing set of cardinality at least z'?

We make use of the following decision problem which is proved to be NP-complete for bipartite graphs and for chordal graphs (see [4]).

OPEN PACKING problem
INSTANCE: A graph $G = \big(V(G), E(G)\big)$ of order n and a positive integer z.
QUESTION: Does G have an open packing set of cardinality at least z?

Theorem 2.1. *The k-TOTAL LIMITED PACKING problem is NP-complete even for bipartite graphs and for chordal graphs.*

Proof. The k-TOTAL LIMITED PACKING problem is a member of NP because checking that a set of vertices is a k-total limited packing set of cardinality at least z' can be done in polynomial time.

In what follows, we show how a polynomial time algorithm for the OPEN PACKING problem could be used to solve the k-TOTAL LIMITED PACKING problem in polynomial time. Let G with $V(G) = \{v_1, \cdots, v_n\}$ be a graph as an instance of the OPEN PACKING problem. We set $G' = G \odot (k-1)K_1$, that is, a graph obtained from G by joining $k-1$ new vertices $u_{i_1}, u_{i_2}, \cdots, u_{i_{k-1}}$ to v_i for each $1 \le i \le n$, and set $z' = n(k-1) + z$. Suppose that B is a $\rho_o(G)$-set. It is readily seen by the construction that $B' = B \cup \{u_{1_1}, u_{1_2}, \cdots, u_{1_{k-1}}, \cdots, u_{n_1}, u_{n_2}, \cdots, u_{n_{k-1}}\}$ is a k-total limited packing set of the graph G'. So, $L_{k,t}(G') \ge |B'| = \rho_o(G) + n(k-1)$.

Assume, conversely, that B' is an $L_{k,t}(G')$-set. If we have $u_{i_j} \notin B'$ for some $1 \le i \le n$ and $1 \le j \le k-1$, and $|N_G(v_i) \cap B'| \le k-1$, then $B' \cup \{u_{i_j}\}$ is a k-total limited packing set in G', which contradicts the maximality of B'. Thus, if $u_{i_j} \notin B'$, then $|N_G(v_i) \cap B'| = k$. Let $v_l \in N_G(v_i) \cap B'$. Then, it can be easily checked that $B'' = (B' \backslash \{v_l\}) \cup \{u_{i_j}\}$ is an $L_{k,t}(G')$-set containing u_{i_j}. Therefore, without loss of generality, we can assume that $\{u_{1_1}, u_{1_2}, \cdots, u_{1_{k-1}}, \cdots, u_{n_1}, u_{n_2}, \cdots, u_{n_{k-1}}\} \subseteq B'$. With this in mind, it is now obvious that $B'' = B' \setminus \{u_{1_1}, u_{1_2}, \cdots, u_{1_{k-1}}, \cdots, u_{n_1}, u_{n_2}, \cdots, u_{n_{k-1}}\}$ is an open packing in G. So, $|B''| = |B'| - n(k-1) \le \rho_o(G)$ and we have that $L_{k,t}(G') \le \rho_o(G) + n(k-1)$.

We now get from $L_{k,t}(G') = \rho_o(G) + n(k-1)$ that $L_{k,t}(G') \ge z'$ if and only if $\rho_o(G) \ge z$. It is known from [4] that the OPEN PACKING problem is NP-complete even for bipartite graphs and for chordal graphs. By the construction, if G is a bipartite (resp. chordal) graph, then $G \odot (k-1)K_1$ is also a bipartite (resp. chordal) graph. Consequently, the k-TOTAL LIMITED PACKING problem is NP-complete even for bipartite graphs and for chordal graphs. $\square$

From the result above, we conclude that the problem of computing the k-total limited packing number is NP-hard, even for some special families of graphs. Consequently, it is desirable to bound this parameter in terms of several graph variables. Several bounds on the k-total limited packing number were given in [5]. We here prove the following theorem.

Theorem 2.2. *Let $G = \big(V(G), E(G)\big)$ be a graph, then for every edge l of $E(G)$*

$$L_{k,t}(G) \le L_{k,t}(G - l) \le L_{k,t}(G) + 2.$$

Moreover, these bounds are tight.

Proof. Any k-total limited packing set of G is also a k-total limited packing set of $G-l$. Thus, $L_{k,t}(G) \le L_{k,t}(G-l)$. If C is a cycle on n vertices, then $L_{2,t}(C) = L_{2,t}(C-l)$ for every edge $l \in E(C)$.

Let now that B be an $L_{k,t}(G-l)$-set, and $l = xy$. If $x, y \in B$, then $B - \{x, y\}$ is a k-total limited packing set of G. Hence, $L_{k,t}(G) \ge |B| - 2$. If $x \in B$ and $y \notin B$, then $B - \{x\}$ is a k-total limited packing set of G, and $L_{k,t}(G) \ge |B| - 1$. If $x, y \notin B$, then B is a k-total limited packing set of G, so $L_{k,t}(G) \ge |B|$. Therefore, $L_{k,t}(G-l) \le L_{k,t}(G) + 2$.

Assume G is a double star $ST(x, y)$. Then, $L_{k,t}(G-l) = L_{k,t}(G) + 2$ for $l = xy$. □

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A note on the lower bounds for the energy of graphs

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Abstract

The energy of graph G, denoted by $\epsilon(G)$, is the sum of the absolute values of its eigenvalues. In this paper, we present some lower bounds for $\epsilon(G)$ in terms of minimum degree.

1 Introduction

Let G be an undirected graph without multiple edges and loops. The adjacency matrix $A(G)$ of a graph G of order n takes the form of an $n \times n$ matrix, denoted as $[a_{ij}]$ where the entries a_{ij} are defined such that a_{ij} equals to 1 if the corresponding vertices v_i and v_j are adjacent, and 0 otherwise. The eigenvalues of G is the eigenvalues of its adjacency matrix $A(G)$. The energy $\epsilon(G)$ of G is defined to be the sum of the absolute values of all eigenvalues of $A(G)$. In this paper, we extend some results related to quadrangle-free and mutually disjoint-quadrangles graphs to unveil new insights into the energy of graphs. The pioneering work by Gutman [1] established a lower bound for the energy of triangle- and quadrangle-free regular graphs, providing a foundation for our exploration. This brings us to Theorem 1, that explains the energy of quadrangle-free graphs, considering certain limits on the graph's maximum degree. Expanding on Theorem 1, we further explore graphs with mutually disjoint quadrangles. Theorem 2 brings in a new viewpoint, showing how disjoint quadrangles influence the minimum energy level of a graph. The conditions under which these lower bounds are achieved offer a deeper understanding of graph structures that optimize energy values.

2 Main results

In this section, we state some of new results on the lower bound of energy of graphs.

*Speaker

subjclass[2010]: 05C50, 15A18
keywords: energy of graph, minimum degree, maximum degree

By [4], we know that if G is a triangle- and quadrangle-free regular graph on n vertices, of degree r, $r > 0$, then

$$\epsilon(G) \geq \frac{nr}{\sqrt{2r-1}}.$$

We extend this result as follows:

Theorem 2.1. *Let G be a quadrangle-free graph of order n with minimum degree $\delta \geq 1$ and maximum degree Δ. If $\Delta \leq 2\delta - 1$, then*

$$\epsilon(G) \geq \frac{n\delta}{\sqrt{2\delta-1}}.$$

Moreover, the equality holds if and only if G is the disjoint union of complete graphs K_2.

Theorem 2.2. *Let G be a graph of order n with minimum degree $\delta \geq 1$ and maximum degree Δ in which all quadrangles are mutually disjoint. If $\Delta \leq 2\delta - 1 + \frac{3}{\delta}$, then*

$$\epsilon(G) \geq \frac{n\delta\sqrt{\delta}}{\sqrt{2\delta^2-\delta+2}}.$$

Moreover, the equality holds if and only if G is the disjoint union of complete bipartite graphs $K_{2,2}$.

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Some results on domination polynomial of neighborhood corona of two graphs

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Abstract

The domination polynomial of a graph G of order n is the polynomial $D(G,x) = \sum_{i=0}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i. The neighbourhood corona of two graphs G_1 and G_2 is denoted by $G_1 \circledast G_2$ and is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and joining the neighbours of the i-th vertex of G_1 to every vertex in the i-th copy of G_2. In this talk, we study the domination polynomial of neighborhood corona of some of graphs.

1 Introduction

Let G be a simple graph. For any vertex $v \in V(G)$, the open neighborhood of v is the set $N(v) = \{u \in V(G) | u \sim v\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subset V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subset V$ is a dominating set if $N[S] = V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of the dominating set in G. The total dominating set is a subset D of V that every vertex of V is adjacent to some vertices of D. The total domination number of G is equal to minimum cardinality of total dominating set in G and denoted by $\gamma_t(G)$. An i-subset of $V(G)$ is a subset of $V(G)$ of cardinality i. Let $\mathcal{D}(G,i)$ be the family of dominating sets of G which are i-subsets and let $d(G,i) = |\mathcal{D}(G,i)|$. The polynomial $D(G,x) = \sum_{i=0}^{n} d(G,i)x^i$ is defined as domination polynomial of G [1].

The domination numbers of graph products have been extensively studied in the literature. In particular, a large number of papers have addressed the domination number of Cartesian products, inspired by the conjecture by V. G. Vizing that $\gamma(G \,\Box\, H) > \gamma(G) \times \gamma(H)$ (see [4] for a survey.)

The domination polynomials of binary graph operations, such as, join and corona has been computed [2]. Also, recurrence formulae and properties of the domination polynomials of families of graphs obtained by various products, has been investigated [3]. A clique cover or partition into cliques of a given graph is a partition of the vertices into cliques, subsets of vertices within which every two vertices are adjacent. Given two graphs G and H, assume that $\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$ is a clique cover of G and U is a subset of

*Speaker

subjclass[2010]: 05C31, 05C60

keywords: Domination polynomial, Dominating set, Total dominating set, Neighborhood corona

$V(H)$. Construct a new graph from G, as follows: for each clique $C_i \in \mathcal{C}$, add a copy of the graph H and join every vertex of C_i to every vertex of U. Let $G^{\mathcal{C}} \star H^U$ denote the new graph. The domination polynomial of the clique cover product $G^{\mathcal{C}} \star H^{V(H)}$ or simply $G^{\mathcal{C}} \star H$ studied in [5]. The following theorem gives the domination polynomial of $G^{\mathcal{C}} \star H^U$.

Theorem 1.1. *[5] For two graphs G and H, let $\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$ be a clique cover of G and $U \subseteq V(H)$. Then*

$$D(G^{\mathcal{C}} \star H^U, x) = \prod_{i=1}^{k} D(H^*, x),$$

where H^* is the subgraph of order $|V(H)| + |C_i|$ in $G^{\mathcal{C}} \star H^U$ obtained by adding a copy of the graph H and joining every vertex of C_i to every vertex of U. Moreover,

$$D(G^{\mathcal{C}} \star H, x) = \prod_{i=1}^{k} \Big[((1+x)^{n_i} - 1)(1+x)^{|V(H)|} + D(H, x) \Big],$$

where n_i is the order of C_i.

If each clique C_i of the clique cover $\mathcal{C}$ is a vertex, then $G^{V(G)} \star H$ is the corona of G and H. So the clique cover product of graphs is a generalization of corona product and hence by Theorem 1.1 we have the following result:

Theorem 1.2. *[2] Let G and H be nonempty graphs of order n and m, respectively. Then*

$$D(G \circ H, x) = (x(1+x)^m + D(H, x))^n$$

The neighbourhood corona of two graphs G_1 and G_2 is denoted by $G_1 \circledast G_2$ and is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and joining the neighbours of the i-th vertex of G_1 to every vertex in the i-th copy of G_2. In this paper we investigate the domination polynomial of neighborhood corona of two certain graphs.

2 Main results

In this section, we state some new results on the domination number of neighborhood corona and domination polynomial of neighborhood corona $K_n \circledast K_1$.

Theorem 2.1. *If G is a connected graph of order $n \geqslant 3$, then $\gamma(G \circledast H) = \gamma_t(G)$.*

Theorem 2.2. *(i) For $n \geqslant 3$, $\gamma(K_n \circledast K_1) = 2$.*

(ii)

$$d(K_n \circledast K_1, i) = \begin{cases} \displaystyle\sum_{k=2}^{i} \binom{n}{k}\binom{n}{i-k} + n\binom{n-1}{i-2}; & \text{if } i = 2, \dots, n, \\ \displaystyle\sum_{k+k'=i} \binom{n}{k}\binom{n}{k'}; & \text{if } i = n+1, \dots, 2n. \end{cases}$$

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Introducing a novel vertex-degree-based topological index

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Abstract

Topological indices are numerical descriptors for study the relationship between the properties and structures of molecules. In this paper, we introduce a novel vertex-degree-based topological index that is called redefined Sombor index and it has a linear correlation with Sombor index.

1 Introduction

In this paper, we consider a simple, connected, undirected graph $G(V, E)$ with a vertex set $V(G)$ and an edge set $E(G)$. For any vertex $x \in V(G)$, $N_x(G)$ represents the set containing all neighbors of x. The degree of x equals the number of its neighbors and is denoted by $d_x(G) = |N_x(G)|$. An edge connecting a vertex of degree x to a vertex of degree y is denoted by (x, y)-edge.

Topological indices are numerical descriptors that remain unchanged despite graph isomorphisms. Gutman introduced the Sombor index, a vertex-degree-based topological index, in [1], defined as follows:

$$SO(G) = \sum_{zw \in E(G)} \sqrt{d_z^2 + d_w^2}.$$

He explored various properties of this index on specific graphs and established both lower and upper bounds for Sombor indices in [2]. Redžepović [6] investigated the chemical applicability of Sombor indices and delved into their predictive and discriminative potentials.

In [3] Gutman et al. introduced the product of Sombor index and the modified Sombor index and computed its main properties. Lower and upper bounds for its product are obtained and the extremal graphs are determined. Kulli in [4] introduced the modified neighborhood Sombor index and the modified neighborhood Sombor exponential of a graph and he also computed the its valu and their corresponding exponentials of some dendrimers and Some properties of its obtained.

Liu in [5] introduced multiplicative Sombor index and some graph transformations which decrease or increase the multiplicative Sombor index. By using these transformations, he determined extremal values

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keywords: Sombor index, Topological index, Redefined

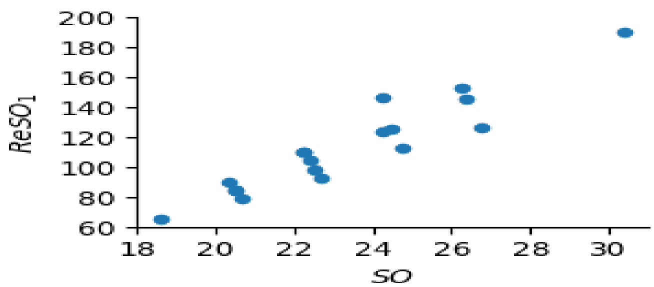

Figure 1: The correlation between the SO and $ReSO_1$ for 18 isomers of Octane.(The correlation coefficient is 0.9419)

of the multiplicative Sombor index of trees and unicyclic graphs. In this paper we introduce a novel vertex-degree-based topological index of graphs that it defined as

$$ReSO_1(G) = \sum_{uv \in E(G)} d_u d_v \sqrt{d_u^2 + d_v^2}, \tag{1}$$

and called redefined Sombor index. According to figure 1 the redefined Somber index $ReSO_1(G)$ and Sombor index $SO(G)$ have a linear correlation.

2 Main results

In this section we present some basic properties of the redefined Sombor index. From definition of $ReSO_1(G)$ in Eq.1 we straightforwardly obtain:

Theorem 2.1. *1. Let G be a $r-$regular graph with n vertices, then $ReSO_1(G) = \frac{nr^4}{2}\sqrt{2}$.*

2. For the cycle graph C_n, we have $ReSO_1(C_n) = 8n\sqrt{2}$.

3. If G be the complete graph K_n, then $ReSO_1(K_n) = \frac{n(n-1)^4}{2}\sqrt{2}$.

4. If Q_k be the hypercube graph with 2^k vertices, then $ReSO_1(Q_k) = \frac{2^k k^4}{2}\sqrt{2}$.

Theorem 2.2. *1. Let $K_{(p,q)}$ be the complete bipartite graph, then $ReSO_1(K_{p,q}) = p^2q^2\sqrt{p^2+q^2}$.*

2. For complete bipartite graph $K_{p,p}$, we have, $ReSO_1(K_{p,p}) = p^5\sqrt{2}$.

3. If S_n be star of order n, then $ReSO_1(S_n) = n^2\sqrt{1+n^2}$.

Theorem 2.3. *Let P_n be the path with n vertices, then $ReSO_1(P_n) = 4(\sqrt{5} + (n-3)\sqrt{8})$.*

Theorem 2.4. *If K_n be the complete graph of order n, and $\overline{K_n}$ be its complement, then for any graph G of order n, we have,*

$$ReSO_1(\overline{K_n}) \leq ReSO_1(G) \leq ReSO_1(K_n).$$

Equality holds if and only if $G \cong K_n$ or $G \cong \overline{K_n}$.

Theorem 2.5. *If $M_1(G)$ be first zagreb index and $SO(G)$ be Sombor index of graph G,then*

$$ReSO_1(G) \leq \frac{M_1^2}{4} SO(G).$$

3 Redefined Sombor index of benzenoid systems

The graph of triangular benzenoid T_p has 6 edges with $(2,2)$, $6p-6$ edges with $(2,3)$ and $\frac{3p(p-1)}{2}$ with $(3,3)$ degree (p is the number of hexagons in the base graph). Therefore

$$ReSO_1(T_p) = 48 + 6(6p-6)\sqrt{13} + \frac{27p(p-1)}{2}\sqrt{18}.$$

The benzenoid rhombus R_p is formed from two copies of a triangular benzenoid T_p by identifying hexagons in one of their base rows. The graph of R_p has 6 edges with $(2,2)$, $8p-8$ edges, with $(2,3)$ and $3p^2-4p+1$ with $(3,3)$ degree. Hence

$$ReSO_1(R_p) = 48 + 6(8p-8)\sqrt{13} + 9(3p^2-4p+1)\sqrt{18}.$$

The benzenoid hourglass X_p is derived from two copies of a triangular benzenoid T_p by overlapping hexagons. The graph of X_p has 8 edges with $(2,2)$ and $12p-16$ edges, with $(2,3)$ and $3p^2-3p+4$ with $(3,3)$ degree. Thus,

$$ReSO_1(X_p) = 64 + 6(12p-16)\sqrt{13} + 9(3p^2-3p+4)\sqrt{18}.$$

In this study, we have introduced the novel vertex-degree-based topological index of graphs that is called redefined Sombor index and determined its value for some certain graphs and some benzenoid systems. Since there exists not much information about this index, it is interesting to test their potential chemical applicability and could be determined the bounds and the exact values of this sombor index from other (chemical) graphs.

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Dominating sets and Sombor index of unicyclic graphs

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Abstract

The unicyclic graphs are great class of chemical structures, in this paper we obtain bound of Sombor index of unicyclic graphs with given domination number.

1 Introduction

Let $G(V, E)$ represent a simple connected and undirected graph having a set of vertices $V(G)$ and a set of edges $E(G)$. For any vertex $w \in V(G)$, $N_w(G)$ denotes the set comprising all neighbors of vertex w. The degree of w equals the number of its neighbors, defined as $d_w(G) = |N_w(G)|$. A vertex with a degree of one is termed a pendent vertex. The graph's diameter refers to the greatest distance between any two vertices within G. A diametral path is a path that contains $D(G)$ edges between two vertices.

A unicyclic graph is a connected graph G, containing exactly one cycle, where $|V(G)| = |E(G)|$. The set $D \subseteq V(G)$ is termed a domination set of G, if for every $w \in V(G)$, either $w \in D$ or there exists $z \in N_w(G)$ such that $z \in D$. The smallest cardinality of D is referred to as the domination number of G, denoted by $\gamma(G)$.

Topological indices are numerical descriptors invariant under graph isomorphisms. Gutman introduced the Sombor index in [3] as a vertex-degree-based topological index, defined by:

$$SO(G) = \sum_{ab \in E(G)} \sqrt{d_a^2 + d_b^2}.$$

He explored several properties of this index concerning specific graphs. Redžepović [5] explored the chemical applicability of Sombor indices, studying their predictive and discriminative potentials, concluding their good predictive potential.

Recently, the Sombor index has been paid attention in mathematics and chemistry studies, especially on trees, unicyclic, and bicyclic graphs. For example, Alidadi et al. [1] obtained the minimum Sombor index for unicyclic graphs with fixed diameter. Cruz and Rada [2] determined the maximum and minimum

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keywords: Dominating set, Unicyclic graphs, Sombor index

Sombor index for unicyclic and bicyclic graphs, while Zhou et al. [8] achieved the extremal Sombor index of trees and unicyclic graphs with given matching number.

Réti et al. [6] established certain bounds on the Sombor index. They demonstrated that among all connected unicyclic graphs of order $n \geq 4$, the cycle graphs C_n possess the minimum Sombor index. They also indicated that the maximal Sombor index is instrumental in determining the classes of all connected graphs with a specific cycle.Sun and Du [7] determined the upper and lower bounds of the Sombor index on trees with a given domination number.

Another famous index based on vertex degrees is the Zagreb index, which has attracted the attention of many researchers, and numerous bounds have been obtained for it using graph parameters, including the domination number. This index is defined as follows:

$$Zg_1 = \sum_{ab \in E(G)} (d_a + d_b).$$

Mojdeh et al. [4] deduced upper bounds of Zagreb indices from unicyclic and bicyclic graphs with domination number, also establishing bounds for the first and second Zagreb indices for trees, unicyclic, and bicyclic graphs with a given total domination number.

In this paper, we present the upper bound of Sombor index of unicyclic graphs with given domination number. In Lemma 1.1, this bound is presented for trees with n vertices and domination number γ.

Lemma 1.1. *[7] If T is a tree with a domination number of γ, then*

$$SO(T) \leq (n - 2\gamma + 1)\sqrt{(n-\gamma)^2 + 1} + (\gamma - 1)\sqrt{(n-\gamma)^2 + 4} + \sqrt{5}(\gamma - 1).$$

The following lemma states that if we remove an edge from a graph, the domination number either remains constant or increases by one unit. We use this lemma to prove Theorem 2.2.

Lemma 1.2. *[4] If G is a connected graph with a domination number $\gamma(G)$, then for the edge e belongs to $E(G)$, $\gamma(G - e) \in \{\gamma(G), \gamma(G) + 1\}$.*

2 Main results

In this section, we will present the upper bound of Sombor index of unicyclic graphs with n vertices and domination number γ. We use Lemma 2.1 to prove Theorem 2.2.

Lemma 2.1. *Let G be a connected graph with $n \geq 3$ vertices and a domination number γ. If we define:*

$$k(n,\gamma) = (n - 2\gamma + 1)\sqrt{(n-\gamma)^2 + 1} + (\gamma - 1)\sqrt{(n-\gamma)^2 + 4} + \sqrt{5}(\gamma - 1), \tag{1}$$

then $k(n,\gamma)$ is a strictly decreasing function with respect to γ.

The following theorem is the main result of this paper, which states an upper bound for unicycle graphs using the domination number. In this case, we remove an edge from the single cycle in the graph and convert it into a tree, and then we get the result using Lemma 1.1.

Theorem 2.2. *If U is a unicyclic graph of order n with domination number $\gamma(U) = \gamma$, and maximum degree Δ. Then,*

$$\begin{aligned} SO(U) \leq\ & (n - 2\gamma + 1)\sqrt{(n-\gamma)^2 + 1} + (\gamma - 1)\sqrt{(n-\gamma)^2 + 4} \\ & + \sqrt{5}(\gamma - 1) + \sqrt{2}\Delta + 2\sqrt{2}(\Delta - 1)^2. \end{aligned}$$

Proof. If U is a unicyclic graph of order n and with domination number γ. If C_1 is the unique cycle of G, and $e = ab \in E(C_1)$ is an edge. Taking the tree $T' = U - e$ alongside Lemma 1.1 suggests that

$$SO(U) = SO(T') + \sqrt{d_a^2 + d_b^2} + \sum_{c \in N_a, c \neq b} \left(\sqrt{d_a^2 + d_c^2} - \sqrt{(d_a - 1)^2 + d_c^2} \right)$$
$$+ \sum_{u \in N_b, u \neq a} \left(\sqrt{d_b^2 + d_u^2} - \sqrt{(d_b - 1)^2 + d_u^2} \right)$$

By Lemma 1.2, we have $\gamma(T') \in \{\gamma, \gamma + 1\}$ and by Lemma 2.1, $k(n, \gamma)$ is strictly decreasing for γ, therefore the result is obtained. □

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Extremal transmission regular and transmission irregular graph with respect to Wiener index

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Abstract

Let G be a simple connected graph. For a vertex u, transmission $Tr_G(u)$ is defined as sum of all distances between u and other vertices. Wiener index of G is sum of distances between all pair of vertices which equals half of sum of transmission of vertices. Wiener complexity is number of different vertex transmission of G. In this paper we study the Wiener index on graphs with maximum and minimum possible value of Wiener complexity.

1 Introduction

All considered graphs are simple and connected. Let $G(V(G), E(G))$ be a graph. Distance between two vertices u and v in G, $d_G(u,v)$ (shortly $d(u,v)$) is the length of shortest path between u and v. Maximum distance from a vertex v is called eccentricity of v and is denoted by $\varepsilon_G(v)$. Center of G, $C(G)$ is set of vertices have minimum eccentricity. Diameter (G) and radius (G) are maximum and minimum eccentricity of vertices of G respectively. Transmission of a vertex v, $Tr_G(v)$ is sum of all distances between v and other vertices of G.

The concept of distance pervades many fields of science as mathematics, and even our daily lives. Distances play a crucial role in location theory and facility location problems, network design in operations research [2, 4], distance-based topological indices in mathematical chemistry, measuring the closeness of groups of individuals in sociology, identifying the role of players in social networks such as the internet, and so on. Transmission also introduces several concepts in graph theory. For example, the well-known topological index, the Wiener index, is defined as half the sum of the vertex transmissions, i.e.

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} Tr(v).$$

The number of distinct vertex transmission in G is called Wiener complexity of G and is denoted by $C_W(G)$. Interesting graph families have been defined based on the transmission recently. For instance,

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keywords: transmission of vertex; Wiener complexity; Transmission regular graph, Transmission irregular graph.

if all vertices of the graph G have a same transmission, G is called transmission regular. A transmission irregular graph is a graph which its vertices have distinct transmission. In the other word, a graph G whose Wiener complexity is minimum i.e. $C_W(G) = 1$ is called a transmission regular graph and if G have maximum Wiener complexity i.e. $C_W(G) = n(G)$ introduced as transmission irregular graph. In this paper we study extremal transmission regular and transmission irregular graphs with respect to the Wiener index. The following useful lemma goes back to[3]

Lemma 1.1. *Let u and v be two adjacent vertices of G. Then $Tr(u) - Tr(v) = n_v - n_u$, where n_u (n_v) is the number of vertices lie closer to u (v) than v (u).*

Theorem 1.2. *Let G be a graph of diameter at most 2. Then G is transmission regular graph if and only if G is regular graph.*

Let $n(G)$ denotes the order of G and u be a vertex of G. By a direct calculation we get

$$Tr(u) = 2(n(G) - 1) - \deg(u)$$

Hence two vertices have a same transmission if and only if are of a same degree.

Note that in graphs of diameter at most 2, the number of distinct degree of vertices equals Wiener complexity.

Corollary 1.3. *If G is a connected strongly regular graph then G is transmission regular.*

Theorem 1.4. *Let G be transmission regular graph. Then G is 2−connected.*

Proof. Suppose on the contrary that u is a cut-vertex of G. Let G_1 be a connected component of $G - u$ with minimum order among all connected components of $G-u$. It is evident $n(G_1) < \frac{n(G)}{2}$. Let $y \in V(G_1)$. By Lemma 1.1

$$Tr(y) - Tr(u) > (n(G) - n(G_1)) - n(G_1) > 0$$

Thus $Tr(y) > Tr(u)$ that is a contradiction. □

Since cycles are transmission regular graph and not 3− connected, the above statement could not be extended to 3−connectivity. Next result shows that cycles get maximum Wiener index among all graphs of the same order.

Theorem 1.5. *If G is a transmission regular graph of order n. Then $\binom{n}{2} \leq W(G) \leq W(C_n)$ and the left equality holds if and only if $G = K_n$ and the right equality holds if and only if $G = C_n$*

Proof. The left inequality follows from the fact that complete graph K_n is transmission regular graph and its Wiener index is minimum among all graphs of order n especially among transmission regular graphs. Let $n(G) = n$ and u be a vertex of G. By Lemma 1.1 G is two connected and then each pair of vertices lie in a common cycle. This yields that $\varepsilon(u) \leq \frac{n}{2}$. More over for each positive integer $t \leq \varepsilon(u) - 1$ there is at least two vertices at distance t from u. Consider two cases: First n is odd number. Then we get

$$Tr(u) \leq 2(1 + 2 \cdots \frac{n-1}{2}) = \frac{1}{4}(n^2 - 1).$$

Consequently

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} Tr(u) \leq \frac{1}{2} n \frac{1}{4}(n^2 - 1) = \frac{n^3 - n}{8}$$

Second: n is even number. Similarly we have

$$Tr(u) \leq 2(1 + 2 \cdots \frac{n}{2} - 1) + \frac{n}{2} = \frac{1}{4}(n^2).$$

Consequently

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} Tr(u) \leq \frac{1}{2} n \frac{1}{4}(n^2) = \frac{n^3}{8}$$

In both cases, the equality holds if and only if all vertices are of eccentricity $\frac{n}{2}$. This yields G is a cycle. Proof is completed. □

Since the distance between vertices is preserved under graph automorphism, set of asymmetric graphs contains irregular graphs. In the sequel, extremal transmission irregular trees with respect to the Wiener index are determined. The tree $T(m,n,k)$ represents a tree consisting of 3 paths of orders m, n, and k, with an isolated vertex u connected to an end vertex of each of the 3 paths.

Theorem 1.6. *If T is an transmission irregular tree of order $n = n(T)$, then*

$$W(T) \leq \frac{1}{6}\left(n^3 - 13n + 48\right).$$

Moreover, equality holds if and only if $T = T_{1,2,n-4}$.

Proof. Let T be a tree that has the maximum Wiener index among all asymmetric trees of order n. Let P be a diametrical path in T. As T is asymmetric, $P \neq T$. Let v be a leaf of T that does not lie on P. From the well known fact that the path P_n has the maximum Wiener index among all graphs of order n (and hence among all trees of the same order), we get

$$W(T) = W(T - v) + Tr(v) \leq W(P_{n-1}) + Tr(v). \tag{1}$$

Since T is asymmetric and v does not lie on the diametrical path P, we have $\mathrm{ecc}(v) \leq n - 3$. (Indeed, $\mathrm{ecc}(v) = n - 2$ would mean that $T = T_{1,1,n-3}$ which has a non-trivial automorphism.) This in turn implies that $Tr(v)$ is largest possible if v is adjacent to the third vertex of P (or the third before last vertex of P for that matter). As T has the maximum possible Wiener index, we must have equality in (1), which implies that $T - v = P_{n-1}$ and v is adjacent to the third (or the before last third vertex) of P_{n-1}, that is, $T = T_{1,1,n-4}$. Finally,

$$W(T_{1,1,n-4}) = W(P_{n-1}) + Tr(v) = \binom{n}{3} + \left(\binom{n-2}{2} + 5\right) = \frac{n^3 - 13n + 48}{6}.$$

□

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Symmetric and Hemi-Cartesian product with application in zero divisor graph

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Abstract

In this paper, we generalized cartesian product of two graphs, hemi-cartesian product, by an arbitrary induced subgraph in which it is not comutative, necessarily. Also, we introduced symmetric product of a graph by itself. Furthermore, the zero divisor graph of 2×2 matrices over a field F are obtained by applying hemi-cartesian, cartesian, tensor and symmetric product and join, A–join and union of some graphs.

1 Introduction

Sabidussi [3, p. 396], has defined the A–join of a set of graphs $\{G_a\}_{a \in A}$ as the graph H with vertex and edge sets

$$\begin{aligned} V(H) &= \{(x,y) \mid x \in V(A) \ \& \ y \in V(G_x)\}, \\ E(H) &= \{(x,y)(x',y') \mid xx' \in E(A) \ \textit{or else} \ x = x' \ \& \ yy' \in E(G_x)\}. \end{aligned}$$

It is clear that when $A = K_2$, the A–join of graphs Γ_1 and Γ_2 is the ordinary join of two graphs.

All zero divisor elements of R is denoted by $Z(R)$. The zero divisor graph of a ring was introduced by Beck in 1988. Following Beck [2], we assume that R is a ring and $G(R)$ is a simple graph with vertex set $V(G(R)) = R$ and edge set $E(G(R)) = \{xy \mid x, y \in R \ \& \ xy = 0\}$. Anderson and Livingston [1], considered the set of all non-zero zero divisors as the vertex set to simplify the Beck's zero divisor graph. The edges are the same as the Beck's seminal paper. In this paper we use the Anderson–Livingston's definition of zero devisor graph and so all rings considered here is not integral. This graph is denoted by $\Gamma(R)$. Moreover, we consider graphs without multiple edges. For a subset A of R, $A^\star$ denotes the set of nonzero elements of A. $UM_2(F)$ is denotes the set of all upper triangular matrices over a field F.

If G is a graph, $A \subseteq V(G)$ and $B \subseteq E(G)$ then $G \smallsetminus (A, B)$ is a graph with vertex set $V(G) \smallsetminus A$ and edge set $E(G) \smallsetminus B$. The one vertex graph with a loop is denoted by K_1^l. Suppose $T \subseteq V(G)$. The induced subgraph $G[T]$ is a subgraph with $V(G[T]) = T$ and $E(G[T]) = \{e = uv \in E(G) \mid \{u, v\} \subseteq T\}$.

*Speaker

subjclass[2010]: 05C76, 13A99, 16U99

keywords: Graph operations, Zero divisor graph, Ring

2 Main results

In this section, we firstly introduce hemi-cartesian and symmetric product, then state some results on zero divisor graph of 2×2 matrices over a field F and show that how one can apply the products to characterized the zero divisor graph.

The hemi-Cartesian product $G \overset{G'}{\boxminus} H$ of G and H related to the induced subgraph G' of G is a graph with the vertex and edge sets

$$\begin{aligned} V(G \overset{G'}{\boxminus} H) &= V((G \smallsetminus G') \square H) \cup V(G'), \\ E(G \overset{G'}{\boxminus} H) &= \{t(x,y) \mid t \in V(G'), (x,y) \in V((G \smallsetminus G') \square H), tx \in E(G)\} \;\cup\; E(G') \\ &\cup\;\; E((G \smallsetminus G') \square H), \end{aligned}$$

respectively.

The symmetric product of a graph G by G, $G \star G$, is a graph with vertex set $V(G \star G) = V(G) \times V(G)$ and edge set $E(G \star G) = \{(a,b)(c,d) \mid ad \in E(G) \text{ or } bc \in E(G)\}$.

For a ring R, the looped zero divisor graph $\Gamma_l(R)$ and looped Beck's zero divisor graph $\Gamma_{B,l}(R)$ are defined as follows:

$$\begin{aligned} V(\Gamma_l(R)) &= Z(R)^\star \;\;,\;\; V(\Gamma_{B,l}(R)) = R, \\ E(\Gamma_l(R)) &= \{xy \mid x,y \in Z(R)^\star \;\&\; (xy = 0 \text{ or } yx = 0)\}, \\ E(\Gamma_{B,l}(R)) &= \{xy \mid x,y \in R \;\&\; (xy = 0 \text{ or } yx = 0)\}. \end{aligned}$$

Theorem 2.1. [4] *Suppose $R_1, \cdots, R_n$ are rings. Then,*

$$\begin{aligned} \Gamma_{B,l}(R_1 \times \cdots \times R_n) &= \Gamma_{B,l}(R_1) \otimes \cdots \otimes \Gamma_{B,l}(R_n) \\ \Gamma(R_1 \times \cdots \times R_n) &= (\Gamma_{B,l}(R_1) \otimes \cdots \otimes \Gamma_{B,l}(R_n)) \smallsetminus (\{0\} \cup \nu, \lambda), \end{aligned}$$

where λ and ν are the set of loops and pendant vertices of $\Gamma_{B,l}(R_1 \times \cdots \times R_n)$, respectively.

Now, consider the wheel graph W_4 with the vertex set $V(W_4) = \{a,b,c,d,e\}$ such that $deg(a) = 4$ and $\{bc, cd, de, eb\} \subseteq E(W_4)$. Define $X'''(F)$ to be the W_4–join of K_1, $\Phi_{F^\star}$, $\Phi_{F^\star}$, K_1 and K_1. We label the vertices of the subgraph $\Phi_{F^\star}$ corresponding to b by b_α and the vertices of the subgraph $\Phi_{F^\star}$ corresponding to c by c_α, where $\alpha \in F^\star$. The vertices a, d and e of W_4 are labeled by the same letters in $X'''(F)$.

We now define the graph $\Delta'''(F)$ as a $X'''(F)$–join of some graphs isomorphic to $K_{F^\star}$ or $\Phi_{F^\star}$ in such a way that for each $\alpha \in F^\star$, the vertices a, b_α, c_α, d and e are corresponding to graphs $K_{F^\star}$, $\Phi_{F^\star}$, $\Phi_{F^\star}$, $\Phi_{F^\star}$ and $\Phi_{F^\star}$, respectively. We label the vertices of the subgraph $K_{F^\star}$ corresponding to a by a_α, for every $\alpha \in F^\star$. The vertices of the subgraph $\Phi_{F^\star}$ corresponding to b_α, $\alpha \in F^\star$, by $b_{\alpha,\beta}$, where $\beta \in F^\star$, and the vertices of the subgraph $\Phi_{F^\star}$ corresponding to $c_\alpha, \alpha \in F^\star$, by $c_{\alpha,\beta}$, where $\beta \in F^\star$, are labeled. In a similar way, the vertices of the subgraph $\Phi_{F^\star}$ corresponding to d and e are labeled by d_α and e_α, respectively. Define $\Delta''',^l(F)$ to be a graph with vertex set $V(\Delta'''(F))$ and edge set $E(\Delta'''(F))$ together with one loop on each vertex a_α, $\alpha \in F^\star$. We label the complete graph K_2 by $\{1,2\}$ and define the induced subgraph $X'''(F)_2 = X'''(F)[\{e,d\}]$.

Set $X''(F) = (X'''(F) \overset{X'''(F)_2}{\boxminus} K_2) \smallsetminus (\varnothing, \{(a,1)(a,2)\})$. Then,

$$V(X''(F)) = \{e, d, (a,i), (b_\alpha, i), (c_\alpha, i) \mid i \in \{1,2\}, \alpha \in F^\star\}.$$

With the following mention, subgraph of the zero divisor graph of 2×2 matrices over fields are made by symmetric product. Suppose $\Delta''(F)$ is a $X''(F)$–join of some graphs isomorphic to $K_{F^\star}$ or $\Phi_{F^\star}$ in such a way that for each i, α, $i \in \{1,2\}$ and $\alpha \in F^\star$, the vertices (a,i), (b_α, i), (c_α, i), d and e are corresponding to graphs $K_{F^\star}$, $\Phi_{F^\star}$, $\Phi_{F^\star}$, $\Phi_{F^\star}$ and $\Phi_{F^\star}$, respectively. We define a graph $X(F)$ and a matching $G_{-1}(F)$ for the complete graph on the vertex set $F^\star$ by $E(G_{-1}(F)) = \{xy \mid xy + 1 = 0\}$.

Let $X'(F) = G_{-1}(F) * G_{-1}(F)$. The graph $\Delta'(F)$ is defined as $X'(F)$–join in such a way that the vertex (α, β) of $X'(F)$ corresponds to the complete graph $K_{F^\star}$, if $\alpha\beta + 1 = 0$, and the vertex (α, β) corresponds to $\Phi_{F^\star}$, otherwise. By applying $X'(F)$ and $X''(F)$, $\Gamma(M_2(F))$ are characterized in [5].

Lemma 2.2. *The graphs $\Delta'(F)$, $\Delta''(F)$ and $\Delta'''(F)$ are isomorphic to induced subgraphs of* $\Gamma(M_2(F))$ *and* $\Delta'''(F) \cong \Gamma(UM_2(F))$ *and* $\Delta''',^{l}(F) \cong \Gamma_l(UM_2(F))$. *So,* $(X'''(F) \overset{X'''(F)_2}{\boxminus} K_2) \smallsetminus (\varnothing, \{(a,1)(a,2)\})$ *and* $G_{-1}(F) * G_{-1}(F)$ *are isomorphic to induced subgraphs of* $\Gamma(M_2(F))$.

Theorem 2.3. [5] *Suppose R is a finite commutative reduced ring with unity and* $H = \otimes_{i=1}^{m}(\Phi_{(q_i-1)^2 q_i} + K_1^l + \Delta''',^{l}(F_i))$, *For some prime powers q_i. Then,* $\Gamma(UM_2(R)) \cong (\otimes_{i=1}^{m}(\Phi_{(q_i-1)^2 q_i} + K_1^l + \Delta''',^{l}(F_i))) \smallsetminus (\{0\} \cup \nu, \lambda)$, *where ν and λ are the set of all pendant vertices and loops of the graph H, respectively.*

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Recent Results on Perfect State Transfer Problem on Cayley graphs

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Abstract

In this paper, we review recent results on the existence of perfect state transfer (PST for short) on Cayley graphs.

1 Introduction

Let Γ be a simple undirected graph with adjacency matrix A. The continuous-time quantum walk on Γ is defined through the time-dependent unitary matrix

$$H(t) = H_A(t) = \exp(-\mathbf{i}tA) = \sum_{k=0}^{\infty} \frac{(-\mathbf{i})^k t^k A^k}{k!}, \quad 0 < t \in \mathbb{R},\ \mathbf{i} = \sqrt{-1},$$

which is known as *transfer matrix* of Γ. This concept first introduced by Farhi and Gutman by a motivation of the Schrödinger equation. They used it as a paradigm to design efficient quantum algorithms.

The significance of the study of quantum state transfer lies in its applications to the theory of Quantum Information and Computation. In fact, one of the cruical ingredients in most of quantum information processing protocols is the transfer of a quantum state from one location to another location. Quantum spin network is an example of physical systems that can serve as a quantum channel. By considering the networks as graphs, in algebraic graph theory, one of the main questions is to find a characterization of graphs having perfect state transfer. We say that a graph Γ has a *perfect state transfer* (PST for short) from the vertex u to the vertex v at time t if the (u,v)-entry of $H(t)$, denoted by $H_{u,v}(t)$, has absolute value 1. If $|H_{u,u}(t)| = 1$ then we say that Γ is periodic at u with period t. Γ is called periodic if it is periodic at all vertices with period t.

Let G be a group and S be a non-empty subset of G. The Cayley graph of G with respect to S, $\mathrm{Cay}(G,S)$, is a graph with vertex set G where g is adjacent to h if $hg^{-1} \in S$. The graph $\mathrm{Cay}(G,S)$ is called quasiabelian if S is a conjugate-closed subset of G. A graph is called integral if all of the eigenvalues of its adjacency matrix are integers. In this paper, we review recent results on PST problem on Cayley graphs including the papers of the author.

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keywords: Perfect state transfer, eigenvalue, Cayley graph

2 Main results

In this section, we state some of recent results on the PST problem on Cayley graphs.

Proposition 2.1. *[1, Corollary 7 and Lemma 11] Let $\Gamma = \mathrm{Cay}(G,S)$ be an undirected Cayley graph over a finite group G with irreducible unitary matrix representations $\varrho^{(1)},\dots,\varrho^{(m)}$. Let d_l be the degree of $\varrho^{(l)}$. For each $l \in \{1,\dots,m\}$, define a $d_l \times d_l$ block matrix $A_l := \varrho^{(l)}(S)$. Let $\chi_{A_l}(\lambda)$ and $\chi_A(\lambda)$ be the characteristic polynomials of A_l and A , respectively. Then*

(1) there exists a basis $\mathcal{B}$ such that $[A]_{\mathcal{B}} = \mathrm{Diag}(A_1 \otimes I_{d_1},\dots,A_m \otimes I_{d_m})$.

(2) $\chi_A(\lambda) = \Pi_{l=1}^m \chi_{A_l}(\lambda)^{d_l}$.

(3) Let $v_{(k)}$ be an eigenvector of A_k, $1 \le k \le m$, associated with λ. Then the following vectors are distinct linearly independent d_k eigenvectors of Γ associated with λ:

$$v^j_{(k)} := \sum_{g\in G}\left[v_{(k)}\cdot \varrho^{(k)}_j(g)\right]e_g, \quad 1\le j\le d_k$$

where $\cdot$ is the usual inner product and $\varrho^{(k)}_j(g)$ is a vector whose coordinates are the coordinates of jth column of $\varrho^{(k)}(g)$.

Corollary 2.2. *Let A be an abelian group, $Irr(A) = \{1 = \chi_1,\dots,\chi_n\}$, $G = Dih(A,x)$ or $G = Dic(A,y,x)$, and $\Gamma = Cay(G,T)$, where $T = T_1 \cup xT_2$ for some $T_1,T_2 \subseteq A$. if $1 \ne g_1^{-1}g_2 \in A$ then Γ has a PST between g_1 and g_2 if and only if the following conditions hold:*

(i) the order of $g_1^{-1}g_2$ is two,

(ii) Γ is integral, and for each i, $\chi_i(T_1)$ and $|\chi_i(T_2)|$ are integers,

(iii) $\nu_2(|T|-\lambda_i^+) = \nu_2(|T|-\lambda_i^-)$ is the same integer, say k, for all i that $\chi_i(g_1^{-1}g_2) = -1$ and for all i with $\chi_i(g_1^{-1}g_2) = 1$, $\nu_2(|T|-\lambda_i^+) > k$ and $\nu_2(|R|+|S|-\lambda_i^-) > k$,

where $\lambda_i^+ = \chi_i(T_1) + |\chi_i(T_2)|$ and $\lambda_i^- = \chi_i(T_1) - |\chi_i(T_2)|$, $i = 1,\dots,n$.

Also, if $g_1^{-1}g_2 \notin A$, then Γ has a PST between g_1 and g_2 at time t if and only if the following conditions hold:

(i) $\chi_j(T_2) \ne 0$ for each j,

(ii) if $g_1 \in A$ and $g_2 \notin A$ then $\chi_j(g_1^{-1}g_2) = \frac{|\chi_j(T_2)|}{\chi_j(T_2)}\exp(-\mathbf{i}(\lambda_1^+ - \lambda_j^+)t)$, and if $g_1 \notin A$ and $g_2 \in A$ then $\chi_j(g_1^{-1}g_2) = \frac{|\chi_j(T_2)|}{\overline{\chi_j(T_2)}}\exp(-\mathbf{i}(\lambda_1^+ - \lambda_j^+)t)$, where in the both cases $\exp(-\mathbf{i}(\lambda_1^+ - \lambda_j^+)t) = \pm 1$.

(ii) $\chi_j(T_1), |\chi_j(T_2)| \in \mathbb{Z}$ for each j, in particular Γ is integral,

(vi) $v_2(|T_2|) = v_2(|\chi_j(T)|)$ for all j.

Also Γ is periodic if and only if Γ is integral. Furthermore, the minimum period of the vertices is $\frac{2\pi}{M}$, where $M = \gcd\{\lambda - \lambda_1^+ \mid \lambda \in Spec(\Gamma) \setminus \{\lambda_1^+\}\}$.

Theorem 2.3. *Let $\mathrm{Cay}(G,R)$ be a quasi-abelian Cayley graph over a group G of order n. Suppose that G has m non-equivalent irreducible representations. For $1 \le k \le m$, let λ_k be an eigenvalue of $\mathrm{Cay}(G,R)$ with multiplicity d_k^2 satisfying $\sum_{k=1}^m d_k^2 = n$. For $g,h \in G$, $\mathrm{Cay}(G,R)$ has PST between g and h at time t if and only if the following hold, where $M = \gcd(|R| - \lambda_k \mid 1 \le k \le m)$ and $\Omega^- = \{k \mid \chi_k(gh^{-1}) = -d_k, 1 \le k \le m\}$.*

1) *$\mathrm{Cay}(G,R)$ is an integral graph.*

2) *For every $1 \le k \le m$, $\chi_k(gh^{-1}) = \pm d_k$.*

3) *There exists an integer μ such that $v_2(|R| - \lambda_k) = \mu$ for $k \in \{1 \le j \le m \mid \chi_j(gh^{-1}) = -d_j\}$ and $v_2(|R| - \lambda_k) \ge \mu + 1$ for $k \in \{1 \le j \le m \mid \chi_j(gh^{-1}) = d_j\}$.*

4) $t \in \begin{cases} \{\frac{2z\pi}{M} \mid z \in \mathbb{Z}\}, & g = h \text{ or for each } k \in \Omega^-, \lambda_k = |R|; \\ \{\frac{(1+2z)\pi}{M} \mid z \in \mathbb{Z}\}, & \text{otherwise.} \end{cases}$

Moreover, the order of gh^{-1} is two and $\mathrm{Cay}(G,R)$ is periodic if and only if it is an integral graph and the period is $t \in \{\frac{2z\pi}{M} \mid z \in \mathbb{Z} \setminus \{0\}\}$.

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Generalzed Splitting and Element Splitting Operation on p-Matroids

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Abstract

In this paper, we define generalizd splitting and element splitting operations on p-matroids.The p-matroids are the matroids representable over GF(p). The circuits and the bases of the new matroid are characterized in terms of circuits and bases of the original matroid, respectively.

1 Introduction

Note that the maximum number of pages for the extended abstract is three pages.

A matroid M is an ordered pair $(E,\mathcal{I})$ consisting of a finite set E and a collection$\mathcal{I}$ of subsets of E having the following three properties:
$(I1)\varnothing \in \mathcal{I}$.
$(I2) If I \in \mathcal{I}$ and$I' \subset I$, then$I' \in \mathcal{I}$.
$(I3) If I_1, I_2$are in$\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 I_1$ such that $\mathrm{I}_1 \cup e \in \mathcal{I}$.
We shall call (I2) and (I3) the hereditary and independence augmentation properties. If M is the matroid $(E,\mathcal{I})$, then M is called a matroid on E. The members of $\mathcal{I}$ are the independent sets of M, and E is the ground set of M. We shall often write $\mathcal{I}$(M) for$\mathcal{I}$ and E(M) for E, particularly when several matroids are being considered.We call a maximal independent set in M a basis or a base of M. The collection of the bases of M denoted by $\mathcal{B}$(M).
A subset of E that is not in $\mathcal{I}$ is called dependent.
A minimal dependent set in an arbitrary matroid M will be called a circuit of M and we shall denote the set of circuits of M by $\mathcal{C}$ or $\mathcal{C}$(M)
Whenever E is the set of edges of graph G and $\mathcal{C}$ is the set of cycles of G, then $\mathcal{C}$ is the set of circuits of a matroid is on E. The matroid derived from the graph G is called the cycle matroid or polygon matroid of G. It is denoted by M(G).
Let A be a matrix with n rows and m columns on the field F and E, the set of labels of the columns of the matrix A, and also $\mathcal{I}$is the set of all subsets of E that are linearly independent in the vector space V(m,F), in this case $(E,\mathcal{I})$ is a metroid. We call this metroid the vector metroid obtained from the matrix A and

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we denote it by M[A]. Let M be a p -matroid on a set E and let T E. Suppose A is a matrix representation of M over the prime field GF(p). Let A_T be the matrix obtained by adjoining an extra row with entries zero everywhere except in the column corresponding to the members of T where it takes the value 1. The vector matroid of matrix A_T is denoted by M_T . The transition from M toM_T is called splitting operation and the matroid M_T is called the splitting matroid. LetA_T be the matrix obtained from A_T by adjoining an extra column labeled z with entries zero everywhere except in the last row where it takes the value 1. The vector matroid of matrix A'_T is denoted by M'_T. The transition from M toM'_T is called element splitting operation and the matroidM'_T is called the element splitting matroid.
Let C be a circuit of M, and $C \cap T \neq \emptyset$. We say C is a PT-circuit of M, if it is a also circuit of M_T. And if C is not a circuit of M_T , we call it an NPT- circuit of M. We use $\mathcal{C}_0$ to denote the collection of P T -circuits and the circuits containing no element of T. Consider subsets of E of the type$C \cup I$ I where $C = \{u_1, u_2, u_3, ..., u_l\}$ is an NP T -circuit of M which is disjoint from an independent set $I = \{v_1, v_2, v_3, ..., v_k\}$ and $T \cap (C \cup I) \neq \emptyset$. We say $C \cup I$ is a PT-dependent set if it contains no member of $\mathcal{C}_0$ and there are non-zero constants $\alpha_1, \alpha_2, ..., \alpha_l$ and $\beta_1, \beta_2, .., \beta_k$ such that
$\sum_{i=1}^{l} \beta_i u_i + \sum_{j=1}^{k} \beta_j v_j = 0$ and $\sum\limits_{x \in T \cap (C \cup I))}$ coeff.(x)$\equiv 0(mod p)$.

2 Main results

Lemma 2.1. [3]
Let M be a p -matroid and $C_1, C_2$are disjoints NPT -circuits of M. Then $C_1 \cup C_2$ is a dependent set in M_T.

Corollary 2.2. *Let C_1 and C_2 be NP T -circuits of M, and I be an independent set of M. Then $C_1 \cup C_2 \cup I$ cannot be a circuit of M_T .*

The above lemma characterizes all the circuit of M_T containing z. We denote the class of such circuits by Cz. Thus $C_z = \{\cup z : CisNPT - circuitofM\}$

Lemma 2.3. *Let C be a circuit of M_T .Then $z \in C$ if and only if C z is an NP T -circuit of M.*

The above theorem describes the circuits of splitting matroids M_T, and the element splitting matroids M'_T in terms of the circuits of the original p -matroids M.

Theorem 2.4. *Let M be a p -matroid on the ground set E and $T \subset E$. Then*

(i) $\mathcal{C}(M_T) = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}$

(ii) $\mathcal{C}(M'_T) = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_z$
where
$\mathcal{C}_0 = \{C \in \mathcal{C}(M)$:*C is a PT-circuit or* $C \cap T = \emptyset\}$
$\mathcal{C}_1$= *minimal elements of* $\{C \cup I : (C \cup I)$ *is a PT- circuit or* $C \cap T = \emptyset\}$
$\mathcal{C}_2$ = *minimal elements of* $\{C_1 \cup C_2$*:* C_1, C_2 *are NPT-circuits,* $C_1 \cap C_2 = \emptyset$ *and*$C_1 \cup C_2$ *contains no member of* $\mathcal{C}_0$*and* $\mathcal{C}_1\}$

The above theorem describes the bases of splitting matroids M_T, and the element splitting matroids M'_T in terms of the bases of the original p -matroids M.

Theorem 2.5. *Let M be a p -matroid, T, and M contains an NPT -circuit. Then*

(i) $\mathcal{B}(M_T) = \mathcal{B}_1 = \{B \cup x : B \in \mathcal{B}(M), x \notin B$ *and* $B \cup x$ *contains neither PT-circuit nor PT-dependent set* $\}$

(ii) $\mathcal{B}(M'_T) = \mathcal{B}_1 \cup \mathcal{B}_z, where \mathcal{B}_z = \{B \cup z : B \in \mathcal{B}(M)\}$.

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The g-good-neighbor diagnosability of triangle-free graphs

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Abstract

In this paper we investigate the g-good-neighbor diagnosability of triangle-free graphs under the MM^* and PMC models. We show that if G is a triangle-free graph with minimum degree δ and does not contain any subgraph isomorphic to $K_{\delta,\delta}$, then G is $(\delta-1)$-diagnosable and also g-good-neighbor δ-diagnosable under MM^*-model, where $g \geq 2$ and $|V(G)| \geq 2\delta+1$. Moreover, if G does not have a subgraph isomorphic to $K_{\delta-1,\delta-1}$ then it is g-good-neighbor $(\delta+1)$-diagnosable under PMC model, for $g \geq 1$.

1 Introduction

A system is said to be *t-diagnosable* if all faulty units can be identified provided the number of faulty units present does not exceed t. The *diagnosability* of a system is the maximal number of faulty processors that the system can guarantee to diagnose.

For the purpose of self-diagnosis of a system, some different models have been proposed. Among the proposed models, the comparison diagnosis model, which is also called MM model, in [1] and the PMC model in [3] are widely used. In the MM model, each processor performs a diagnosis by sending the same inputs to each pair of its distinct neighbors and then compares their responses. The result of a comparison is either that the two responses agree or disagree. Based on the results of all the comparisons, one needs to decide the faulty or non-faulty (fault-free) status of the processors in the system. The MM^* model was first proposed by Sengupta and Dahbura [4] which is modification of the MM model. In this model any processor has to test every pair of its adjacent processors. In the PMC model it is assumed that a processor can test the faulty or fault-free status of another adjacent processor. Under the PMC model, only processors with direct link are allowed to test each other. In both models, it is assumed that if a processor is fault-free, it should always give correct and reliable test results and if a processor is faulty, then its test results may be correct or incorrect. In 2012, Peng et al. [2] proposed a measure for fault diagnosis of the system, namely, the g-good-neighbor diagnosability, which requires that every fault-free node has at least g fault-free neighbors.

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2 Preliminaries

In the MM model, a self-diagnosable system of a graph X is often represented by a multigraph $M(V,L)$, where V and L are the vertex set of X and the labeled edge set, respectively. $(u,v;w)$ is defined as a labeled edge, if vertices u and v are adjacent to w, which implies that u and v are being compared by w. Since a pair of vertices may be compared by different vertices, M is a multigraph. For $(u,v;w) \in L$, we use $\sigma((u,v;w))$ to denote the result of comparing vertices u and v by w. For w being fault-free, if both u and v are fault-free, then $\sigma((u,v;w)) = 0$, otherwise $\sigma((u,v;w)) = 1$. If w is faulty, $\sigma((u,v;w))$ may be either 0 or 1, which implies the result is unreliable. The collection of all comparison results in $M(V,L)$ defined as a function $\sigma : L \mapsto \{0,1\}$, is the *syndrome* of the diagnosis.

In the PMC model, a self-diagnosable system of a graph G is often represented by a digraph $D(V,L)$, where V and L are the vertex set of G and the order edge set, respectively. If vertex u is adjacent to v, (u,v) is defined as a directed edge , which implies that u can test v. For $(u,v) \in L$, we use $\sigma((u,v))$ to denote the result of testing vertex v by u. For u being fault-free, if v is fault-free, then $\sigma((u,v)) = 1$; otherwise $\sigma((u,v)) = 0$. If u is faulty, $\sigma((u,v))$ may be either 1 or 0, which implies the result is unreliable. The collection of all comparison results in $D(V,L)$ defined as a function $\sigma : L \mapsto \{0,1\}$, is the syndrome of the diagnosis. This study assumes that each node u tests the other whenever they are adjacent to it.

For a given syndrome σ, a fault set F of processors in the system is called to be *compatible* with the syndrome σ, if the syndrome can arise from the circumstance that all vertices in F are faulty while all vertices in $V(G) - F$ are fault-free. A faulty comparator can lead to unreliable results, so a set of faulty vertices may produce different syndromes. Let $\sigma_F = \{\sigma \mid \sigma \text{ is compatible with } F\}$. Two distinct subsets F_1 and F_2 of $V(X)$ are said to be *indistinguishable* if and only if $\sigma_{F_1} \cap \sigma_{F_2} \neq \varnothing$; otherwise, F_1, F_2 are *distinguishable*.

A faulty set $F \subset V$ is called a g-good-neighbor faulty set if $|N(v) \cap (V - F)| \geq g$ for every vertex $v \in V - F$. A g-good-neighbor cut of a graph G is a g-good-neighbor faulty set F such that $G - F$ is disconnected. Suppose that F_1 and F_2 are two distinct g-good-neighbour faulty subsets of G, with $|F_1| \leq t$, $|F_2| \leq t$, G is called g-good-neighbour t-diagnosable if and only if F_1 and F_2 are distinguishable for any distinct pair of (F_1, F_2).

3 Main results

Lemma 3.1. *Any triangle-free graph $G = (V,E)$ with $|V| \geq 2\delta$ is $(\delta - 2)$-diagnosable.*

Lemma 3.2. *The complete bipartite graph $K_{n,n} (n \geq 2)$ is not $(n-1)$-diagnosable.*

Theorem 3.3. *Let $G = (V,E)$ be a triangle-free graph with $|V| \geq 2\delta$. Then G is $(\delta-1)$-diagnosable under the MM^* model if G has not subgraph isomorphic to $K_{\delta,\delta}$, otherwise G is $(\delta - 2)$-diagnosable.*

Lemma 3.4. *Any triangle-free graph $G = (V,E)$ with $|V| \geq 2\delta$ is g-good neighbor $(\delta - 1)$-diagnosable for $g \geq 2$.*

Lemma 3.5. *The complete bipartite graph $K_{n,n}$ is not g-good neighbor n-diagnosable, for any $1 \leq g \leq \frac{n}{2}$.*

Theorem 3.6. *Let $G = (V,E)$ be a triangle-free graph with $|V| \geq 2\delta + 1$. Then G is g-good neighbor δ-diagnosable (for $g \geq 2$) if G has not subgraph isomorphic to $K_{\delta,\delta}$, otherwise G is g-good neighbor $(\delta-1)$-diagnosable.*

Theorem 3.7. *Let $G = (V,E)$ be a connected triangle-free graph with $|V(G)| > 2\delta + 1$ and $\delta(G) \geq 3$. Then G is g-good neighbor $(\delta+1)$-diagnosable (for $g \geq 1$) if G has not subgraph isomorphic to $K_{\delta-1,\delta-1}$.*

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An novel method for computing dot product dimension of certain disconnected graphs with efficient search

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Abstract

A graph $G = (V(G), E(G))$ is defined as a k-dot product graph when there exists a function $f : V(G) \longrightarrow \mathbb{R}^k$ such that $f(u).f(v) \geq 1$ for any two adjacent vertices u and v. The dot product dimension $\rho(G)$ of G denotes the minimum value k in which G is a k-dot product graph. In this paper, we present an efficient algorithm for characterization of dot product dimension of disconnected graphs with one cycle, where the length of the cycle is greater than or equal to 5.

1 Introduction

Let N denote a social network in which a user is a friend with some other users.

Two users are considered friends if and only if their features are similar. We consider a vector representation of the network N. We denote the feature vector of a user u as $\boldsymbol{u}$, where $\boldsymbol{u}$ is a vector of k features in k dimensions. Let $M = \{\boldsymbol{u} | u \in V\}$. In this paper, we consider a mathematical model known as the dot product model. According to this model, two users, denoted as u and v, are considered friends if and only if $\boldsymbol{u}.\boldsymbol{v} \geq t$, for some fixed, positive threshold t. Let G be a graph associated with N where each vertex represents some user and each edge specifies a friendship between the two endpoints. M with a given threshold t is called a k-dot product representation of G, and G is called a k-dot product graph, where each edge uv satisfies $\boldsymbol{u}.\boldsymbol{v} \geq t$.

Graphs with dot product dimension one can be identified in polynomial time [4]. Kang and Mller proved that recognizing graphs of any fixed dot product dimension $k \geq 2$ is NP-hard [5]. Fiduccia *et al.* examined the dot product dimension of complete multipartite, bipartite, interval graphs and trees [4]. The dot product dimension of unicyclic graphs and connected graphs with at most two cycles are characterized [2, 3]. In our recent paper, we classified the dot product dimension of disconnected graphs with one cycle [1].

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subjclass[2010]: 05C62, 05C85

keywords: k-dot product representation, k-dot product dimension and k-dot product graph.

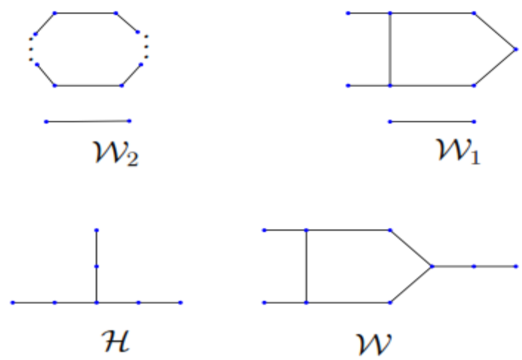

Figure 1: The graphs $\mathcal{H}$, $\mathcal{W}$, $\mathcal{W}_1$ and $\mathcal{W}_2$ of Theorem 2.2.

In this paper, we obtain an efficient algorithm that specifies the dot product dimension of disconnected graphs that have one cycle with length greater than or equal to five.

2 Main results

In this section, we state some of new results on the dot product dimension.

2.1 Some Facts

Definition 2.1. *$\mathcal{W}_2$ is a disconnected graph that is obtained from two components C_n, $n \geq 6$, and P_2 (the graph depicted in Figure 1).*

Theorem 2.2. [1, 2] *If G is $\mathcal{H}$, $\mathcal{W}$, $\mathcal{W}_1$ or $\mathcal{W}_2$ (the graphs depicted in Figure 1), then $\rho(G) = 3$.*

Theorem 2.3. [1] *Let G be a disconnected graph with one cycle that has length greater than or equal to 5. Then we have the following statements.*

(I) $\rho(G) = 3$, if G contains one of the graphs depicted in Figure 1 as an induced subgraph, and

(II) $\rho(G) = 2$, otherwise.

2.2 Our Algorithms

Given a disconnected graph G, we use Depth-First Search (DFS) to efficiently count the number of cycles and components in the graph and it take $O(V + E)$ time. Now we give Algorithm 1 to compute the dot product dimension of G, which is based on Theorems 2.2 and 2.3. We assume that the graph is represented as a data structure-adjacency list.

Algorithm 1: Main Algorithm

Input G, A, B, C
Output dot product dimension of G: ρ
Legend C is the cycle in G
Legend A is the component that contains C
Legend $B = G/A$
if $|A| = 5$ **then**
    ρ =AEQUAL (G,A,B,C)
else if $|A| \geq 6$ **then**
    ρ = BEQUAL (G,B,C)

Next, we explain the details of the subfunctions called in Algorithm 1.

AEQUAL(G, A, B, C): Starting from each vertex s in the cycle, denoted by A, we run Breadth First Search (BFS), to compute the distance of each vertex a from s in the Breadth First tree, and we denote by $d.a$; note that $d.s = 0$. This takes $O(V + E)$ time, where V is the number of vertices and E is the number of edges. Since G is an outerplanar graph, the time complexity is $O(V)$. Lines 3 to 4, lines 5 to 13, and lines 14 to 17 of Algorithm 2 detect the induced subgraph $\mathcal{W}$ depicted in Figure 1, the induced subgraph $\mathcal{H}$ depicted in Figure 1, the induced subgraph $\mathcal{W}_1$ depicted in Figure 1, respectively. Algorithm 2 takes $O(E)$ time. Since G is an outerplanar graph, the time-complexity of Algorithm 2 is $O(|V|)$. For other cases (lines 18 to 19), the algorithm returns 2; this takes $\Theta(1)$ time.

Algorithm 2: AEQUAL(G, A, B, C)

1 **Output** dot product dimension of G: ρ
2 **Legend** $|C| = 5$
3 **if** $\exists\ u, v, c \in C\ s.t\ deg[u], deg[v], deg[c] \geq 3$ && $v \in adj[u]$, $distance(c, v) = distance(c, u) = 2$ && $\exists\ l \in V$ s.t $deg[l] \geq 2$ && $l \in adj[c] \backslash \{C\}$ **then**
4 **return** 3
5 **for** *each* $s \in C$ **do**
6 **if** $\exists\ u, v \in C\ s.t\ deg[u], deg[v] \geq 3$, $v \in adj[u]$ && $\exists\ c \in adj[u]\ \ s.t\ \ c \notin C, deg[c] \geq 2$ **then**
7 **return** 3;
8 **for** *each* $s \in C$ **do**
9 **if** $\exists\ a \in V(A)\ s.t\ \ a.d = l$ && $\exists\ b, c \in adj[a]/C\ s.t\ b.d = c.d = l + 1$ && $deg[b], deg[c] \geq 2$ **then**
10 **return** 3;
11 **for** *each* $s \in B$ **do**
12 **if** $\exists\ a, b, c \in adj[s]\ s.t\ \ deg[a], deg[b], deg[c] \geq 2$ **then**
13 **return** 3;
14 **for** any two adjacent verices u , v in C **do**
15 **if** $deg[u], deg[v] \geq 3$ **then**
16 **if** $\exists\ v \in V(B)\ s.t\ deg[v] \geq 1$ **then**
17 **return** 3
18 **else**
19 **return** 2**;**

BEQUAL(G, B, C): Lines 3 to 8, and lines 9 to 10 of Algorithm 2 detect the induced subgraph $\mathcal{H}$ depicted in Figure 1, the induced subgraph $\mathcal{W}_2$ depicted in Figure 1, respectively. Algorithm 2 takes $O(E)$ time. Since G is an outerplanar graph, the time-complexity of Algorithm 2 is $O(|V|)$. For other cases (lines 11 to 12), the algorithm returns 2; this takes $\Theta(1)$ time.

Algorithm 3: BEQUAL(G, B, C)

1 **Output** dot product dimension of G: ρ
2 **Legend** $|C| \geq 6$
3 **for** each $s \in V(C)$ **do**
4 **if** $\exists\ v \in V(G)\ s.t\ v \notin V(C),\ v \in adj[s]\ and\ deg[v] \geq 2$ **then**
5 **return** 3;
6 **for** *each* $s \in B$ **do**
7 **if** $\exists\ a, b, c \in adj[s]\ s.t\ deg[a], deg[b], deg[c] \geq 2$ **then**
8 **return** 3;
9 **else if** $\exists\ v \in V(B)\ s.t\ deg[v] \geq 1$ **then**
10 **return** 3;
11 **else**
12 **return** 2;

2.3 Time-Complexity

According to the time-complexity of Algorithms 2 to 3, Algorithm 1 takes time $O(|V|)$.

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Mixed partitions of a set and Stirling numbers of the second kind

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Abstract

In this note, we review some results and investigate mixed partitions with extra condition on the sizes of the blocks. We find some recurrence relations and a connection to r-Stirling numbers.

1 Introduction

Set partitions are fundamental and well studied combinatorial objects. Partitions of the set $[n] = \{1, 2, \ldots, n\}$ into k non-empty unlabeled blocks are enumerated by the Stirling numbers of the second kind, also called sometimes set-partition numbers, denoted by $\left\{ {n \atop k} \right\}$. Stirling numbers of the second kind are sometimes introduced by the fundamental recurrence relation:

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

with the initial values $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0$.

The following classical problem was considered in [4]:

Problem 1.1. *Consider $b_1 + b_2 + \cdots + b_n$ balls with b_1 balls labeled by 1, b_2 balls labeled by 2, ..., b_n balls labeled by n and $c_1 + c_2 + \cdots + c_k$ cells with c_1 cells labeled by 1, c_2 cells labeled by 2, ..., c_k cells labeled by k. Evaluate the number of ways to partition the set of these balls into cells of these types.*

The authors in [4] derived some interesting results about the special case $b_1 = b_2 = \cdots = b_n = 1$ and $c_1 = r$, $c_2 = \cdots = c_k = 1$ where n, k and r are positive integers. For the number of partitions of this type, the notation $S(n, k, r)$ was used. We call these numbers mixed Stirling numbers of the second kind. Also in [2] The r-Stirling numbers, denoted by $\left\{ {n \atop k} \right\}_r$, were introduced as the number of partitions of an n-element set into k non-empty subsets such that the first r elements are in distinct subsets.

It is proved that

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*Speaker

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keywords: Set partition, Stirling number, Mixed partition of a set

Theorem 1.2. *[4] For positive integers n, k and r we have*

$$S(n,k,r) = \Sigma_{l=r}^{n-k+1}\binom{n}{l}\left\{{l \atop r}\right\}\left\{{n-l \atop k+1}\right\}(k-1)!.$$

It is also shown that

Theorem 1.3. *[1] For positive integers n, k and r with $r \leq k \leq n$, we have*

$$\left\{{n \atop k}\right\}_r = \Sigma_{i=0}^{k}\binom{r}{i}S(n-1,i+1,k-1)$$

All standard notations in this note may be found in [3].

2 Main results

In the next theorem, we consider another special case $b_1 = b_2 = \cdots = b_n = 1$ and $c_1 = t$, $c_2 = \cdots = c_k = 1$ such that the first r balls are in distinct cells, where n, k, r and t are positive integers and denote this number by $F(n,k,t)_r$.

Theorem 2.1. *For positive integers n, k, r and t we have*

$$F(n,k,t)_r = \Sigma_{i=0}^{\min r,t}\Sigma_{k-1+r-t}^{n-t}\binom{n-t}{l}\binom{t}{i}\binom{k-1}{t-i}(t-i)!(n-t-l)^t S(l,k-1+r-t,r-i)$$

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On the regularity of some binomial ideals associated to some specific graphs

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Abstract

We derive an enhanced upper bound for the regularity of the parity binomial edge ideal of a class of unicyclic graphs. We also determine the exact value of the regularity of binomial edge ideal of a generalization of caterpillar graphs considered in [1].

1 Introduction and Preliminaries

Let G be a simple graph with the vertex set $[n] = \{1, \dots, n\}$ and the edge set $E(G)$. Denoted by K_n, we mean the complete graph on n vertices. For a subset $W \subseteq V(G)$, $G[W]$ represents the induced subgraph of G on the vertex set W, i.e. for $i, j \in W$, one has $\{i, j\} \in E(G[W])$ if and only if $\{i, j\} \in E(G)$. In particular, $G \backslash v$ is the induced subgraph of G on the vertex set $V(G) \backslash \{v\}$. A vertex $v \in V(G)$ is a cut vertex of G whose deletion increases the number of connected components in G. A subset U of $V(G)$ forms a clique if the induced subgraph $G[U]$ is a complete graph. Furthermore, a vertex v is called a free vertex if it belongs to only one maximal clique; otherwise, it is called an internal vertex. Here, $\mathrm{iv}(G)$ denotes the number of internal vertices in G. The neighborhood of a vertex v in G, denoted by $N_G(v)$, is the set of vertices adjacent to v. Given vertex v, G_v is the graph on the vertex set $V(G)$, and the edge set $E(G_v) = E(G) \cup \{\{u, w\} : u, w \in N_G(v)\}$. For an edge e in G, $G \backslash e$ represents the graph with the vertex set $V(G)$ and the edge set $E(G) \backslash \{e\}$. Let $u, v \in V(G)$ such that $e = \{u, v\} \notin E(G)$. Then, G_e denotes the graph with the vertex set $V(G)$ and the edge set $E(G_e) = E(G) \cup \{\{x, y\} : x, y \in N_G(u) \ \ or \ \ x, y \in N_G(v)\}$. A cycle on n vertices is denoted by C_n. A graph is a unicyclic graph if it has precisely one cycle. An odd unicyclic graph is a unicyclic graph with an odd cycle.

Now, let us review some essential notations from commutative algebra. For any homogeneous ideal I in the polynomial ring $R = \mathbb{K}[x_1, \dots, x_m]$, there exists a graded minimal free resolution

$$0 \to \bigoplus_j R(-j)^{\beta_{p,j}(R/I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{1,j}(R/I)} \to \bigoplus_j R(-j)^{\beta_{0,j}(R/I)} \to R/I \to 0,$$

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subjclass[2010]: 05E40

keywords: Parity binomial edge ideal, Binomial edge ideal, Regularity

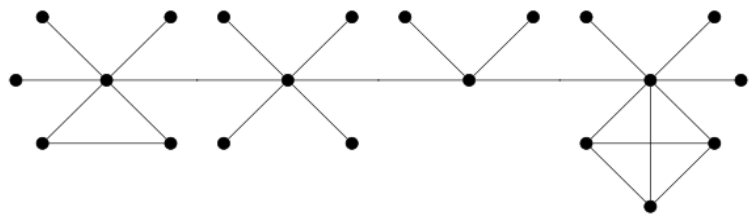

Figure 1: A closed caterpillar

where $R(-j)$ is derived from R by a degree shift of j and $\beta_{i,j}(R/I)$, representing the (i,j)-th graded Betti number of R/I, corresponds to the number of minimal generators of degree j in the i-th syzygy module of R/I. The regularity of R/I, denoted by $\operatorname{reg}(R/I)$, is defined as

$$\operatorname{reg}(R/I) = \max\{j - i \ : \ \beta_{i,j}(R/I) \neq 0\}.$$

Now, we introduce two types of ideals associated with graphs, known as the binomial edge ideal and the parity binomial edge ideal, which were initially introduced by Herzog et al. in [2] and independently by Ohtani in [7], and Kahle et al. in [5].

Definition 1.1. *Let G be a simple graph on the vertex set $[n]$, i.e. G has no loops and no multiple edges. Furthermore, let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring in $2n$ variables. For $i < j$, we set*

$$f_{ij} = x_i y_j - x_j y_i \ , \qquad g_{ij} = x_i x_j - y_i y_j .$$

The binomial edge ideal of G is defined as

$$J_G = \big(f_{ij} \ : \ \{i,j\} \in E(G)\big) \subset S.$$

The parity binomial edge ideal of G is defined as

$$\mathcal{I}_G = \big(g_{ij} \ : \ \{i,j\} \in E(G)\big) \subset S.$$

2 Main result

In this section, we state some results on a conjecture due to Kahle and Krüsemann. In [4], Kahle and Krüsemann proposed a conjecture stating that for a connected graph G on the vertex set $[n]$, one has $l(G) \leq \operatorname{reg}(S/\mathcal{I}_G) \leq n$, where $l(G)$ represents the length of the longest induced path in G. In [6], Kumar established the validity of this conjecture for some classes of graphs. Additionally, Kumar classified those graphs for which the regularity of their parity binomial edge ideal is 3. Hoang and Kahle computed the precise value of the regularity of parity binomial edge ideals of complete graphs [3].

A caterpillar tree is defined as a tree T that possesses a path P, where every vertex of T is either a vertex of P or is adjacent to a vertex of P. Let u and v denote the initial and terminal vertices of the path P, respectively. Let G be the graph obtained by gluing a complete graph and any number of leaves to u as well as v. We call the graph G a closed caterpillar graph (see Figure 1).

Proposition 2.1. *Let G be a closed caterpillar graph. Then*

$$reg(\frac{S}{J_G}) = l(G).$$

Theorem 2.2. *Let G be a unicyclic graph obtained by gluing some paths of any length to some of the vertices of an odd cycle. Then*

$$reg(\frac{S}{\mathcal{I}_G}) \leq |V(G)| - t(G),$$

where $t(G)$ is the number of leaves of G.

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Saturation number of fullerene graphs

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Abstract

The saturation number of a graph G is the cardinality of any smallest maximal matching of G, and it is denoted by $S(G)$. Finding the saturation number of a graph is an NP-hard problem in general, but it is polynomial time solvable for some classes of graphs. In this paper, we present some bounds of saturation number in fullerene graphs in terms of the number of vertices and the diameter of the graph.

1 Introduction

Let $G = (V, E)$ be a simple graph. A **matching** in G is a subset M of E such that no two edges of M have a vertex in common. If every vertex $v \in V$ is incident with an edge $e \in M$, we say that the matching M is **perfect**.

A matching M in G is **maximal** if for every $e \in E \setminus M$, the set $M \cup e$ is not a matching. In other words, a matching M is maximal if it is not a subset of some other matching in G. The saturation number of G is the cardinality of any smallest maximal matching of G, and it is denoted by $S(G)$.

Fullerenes are polyhedral molecules made entirely of carbon atoms.

A **fullerene graph** is a 3-connected 3-regular planar graph with only pentagonal and hexagonal faces. By Euler's furmula, it follows that the number of pentagonal faces is always twelve. Grubaum and Matzkin [5] showed that fullerene graphs with n vertices exist for all $n \geqslant 24$ and for $n = 20$, i.e., there exists a fullerene graph with α hexagons where α is any integer distinct from 1. Although the number of pentagonal faces is negligible compared to the number of hexagonal faces, their layout is crucial for the shape of a fullerene graph. If all pentagonal faces are equally distributed, i.e. their centers are vertices of regular icosahedra, we obtain fullerene graphs of spherical shape with icosahedral symmetry, whose smallest representative is dodecahedron. On the other hand, there is a class of fullerene graphs of tubular shapes, calld **nanotubes**.

2 Main results

In this section, we state some of results on the bounds of saturation number of fullerene graphs.

*Speaker

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keywords: Fullerene graphs, Maximal matching, Saturation number

Proposition 2.1. [3] *let G be a fullerene graph on n vertices. Then*

$$\lceil \frac{n}{4} + 1 \rceil \leqslant S(G) \leqslant \frac{n}{2} - 2 \tag{1}$$

The only property of fullerene graphs used to establish the bounds of proposition 2.1 was their 2-extandibility. A graph G on $n \geqslant 2(n+1)$ vertices is n-extendable if it contains a set of n-independent edges and if any such set can be extended to a perfect matching in G. The lower bound follows directly from theorem 4.3 of [6], while the upper bound was derived considering certain nice subgraphs of fullerene graphs (Subgrapg G' of any graph G is *nice* if $G - V(G')$ has a perfect matching.). it turns out that using another property of fullerene graphs, their 3-regularity, yields a better lower bound on $S(G)$.

Proposition 2.2. [7] *let G be a d-regular graph. Then the size of any maximal matching in G is at most* $(2 - \frac{1}{d})S(G)$.

By the proposition 2.2 we obtain a lower bound on saturation number of fullerene graphs:

Theorem 2.3. [4] let G be a fullerene graph on n vertices. Then $S(G) \geqslant \frac{3n}{10}$.

The following result shows that the regularity of fullerene graphs in fact provides quite good lower bound, and is much more important for their saturation number than 2-extendibility.

Theorem 2.4. [4] For each even integer $n \geqslant 24$ there is a fullerene graph G on n vertices such that $S(G) \leqslant \lceil \frac{n}{3} \rceil$.

Andova in [2] improve the lower bound on the saturation number of fullerene graphs:

Theorem 2.5. [2] Let G be a fullerene graph on n vertices. Then

$$\frac{n}{3} - 2 \leqslant S(G) \leqslant \frac{n}{3} + o(n). \tag{2}$$

The lower bound in last theorem turns out to be tight. There are infinitely many fullerene graphs with the saturation number equel to $\frac{n}{3} - 2$: for example, an (8,0) nanotube with $3k+1$ rings of hexagons and with caps depicted in Figure 1 has $48k + 60$ vertices and admits a maximal matching of size $16k + 18$.

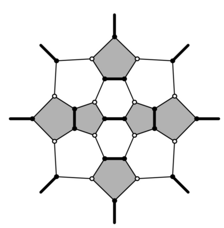

Figure 1: A cap of an (8, 0)-nanotube with saturation number $\frac{n}{3}$

By comparing the lower bound above ($\frac{n}{3} - 2$) and bound in theorem 2.3, ($\frac{3n}{10}$), we find that a fullerene graph can only admit a maximal matching of size exactly $\frac{3n}{10}$ if it has at most 60 vertices. This can occur for fullerene graphs having exactly 20, 30, 40, 50 or 60 (Figure 2).

The distance between two vertices $u, v \in V(G)$ in a connected graph G is the length of any shortest path between these vertices, and it is denoted by $d(u,v)$. A diameter of connected graph G, **diam G**, is the maximum distance between two vertices of G, i.e., $diam(G) = max\{d(u,v) | u, v \in V(G)\}$.

Andova in [1] establish lower and upper bounds for the diameter of fullerene gaphs and use the results to improve the upper bound on their saturation number.

Theorem 2.6. [1] let G be a fullerene graph with n vertices. Then,

$$S(G) \leqslant \frac{n}{2} - \frac{1}{4}(diam(G) - 2), \tag{3}$$

In particular,

$$S(G) \leqslant \frac{n}{2} - \frac{\sqrt{24n - 14} - 15}{24}. \tag{4}$$

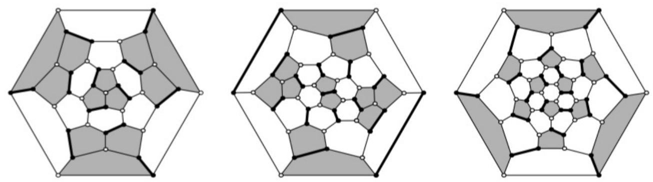

Figure 2: Fullerenes on $n = 40, 50, 60$ with saturation number $\frac{3n}{10}$

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Nil clean graphs over $\mathbb{Z}_p$ rings

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Abstract

Let R be a commutative ring with identity and $N.C(R)$ be the set of all nil clean elements of R. In this paper, we examine the nil clean graph of R, denoted by $G_N(R)$, that vertices of $G_N(R)$ are all nonzero elements of R and two distinct vertices x and y are adjacent if and only if $xy \in N.C(R)$. We investigate the properties of the nil clean graph $G_N(R)$ where R is a direct product of $\mathbb{Z}_{p_i}$'s for some prime numbers $p_1, p_2, \cdots, p_t$. We obtain some graph theoretic properties of the nil clean graph like diameter, girth, clique number, chromatic number.

1 Introduction

The association of a graph with an algebraic object is a dynamic topic in algebraic graph theory. There are many articles about assigning a graph to a ring. Since the idempotents, nilpotents and unit elements of a ring are the main tools for recognizing the structure of the ring, various definitions of graphs related to rings have been given using the concepts. These concepts each characterize the ring structure in many ways. In these structures, the condition of frequently connected vertices is that the sum, difference, or product of the vertices can be zero divisors, a unit, ora nilpotent element, for example, zero divisor graphs [3], unitary Cayley graph [2], a kind of graph structure [5]

Introducing the concept of nil clean elements in [8], The elements are written as the sum of an idempotent element and a nilpotent element, creating a broad field of interesting topics to research, leading to many fascinating topics for occasion see, [4, 6, 7, 9].

Now, let us recall some standard terminology and notations which will be used in this paper. Throughout, unless specially stated, R will be a commutative ring with unity and as usual the rings of integers modulo n will be denoted by $\mathbb{Z}_n$. If $a = e + n$, where e is an idempotent and n is a nilpotent of R, then a is called nil clean. By $Id(R)$, $Nil(R)$ and $U(R)$, we mean the sets of idempotent, nilpotent and unit elements of R, respectively. A ring R is said to be nil clean if each element of R is nil clean. The set of nil clean elements of R is denoted by $N.C(R)$. We associate a (simple) graph $G_N(R)$ for ring R with

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keywords: Nil clean graph, $\mathbb{Z}_n$ rings, diameter, girth, clique number, chromatic number

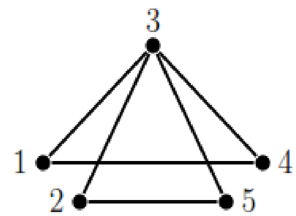

Figure 1: $G_N(\mathbb{Z}_6)$

vertices $V(G_N(R)) = R - \{0\}$, and for distinct $x, y \in R - \{0\}$, the vertices x and y are adjacent if and only if $xy \in N.C(R)$.

2 Main results

The focus of this section will be on elucidating the concept of the nil clean graph of a commutative ring and emphasizing its notable characteristics.

Definition 2.1. *The nil clean graph of a ring R, denoted by $G_N(R)$ is defined by vertices $V(G_N(R)) = R - \{0\}$, and for distinct $x, y \in R - \{0\}$, the vertices x and y are adjacent if $xy \in N.C(R)$.*

As defined, the "nil clean graph" extends beyond the idea of the zero divisor graph. As an example, the nil clean graph $G_N(\mathbb{Z}_6)$ is shown below:
It is easy to see that $Nil(\mathbb{Z}_6) = \{0\}$ and $Id(\mathbb{Z}_6) = \{0, 1, 3, 4\}$, and so $N.C(\mathbb{Z}_6) = Id(\mathbb{Z}_6)$.

A complete graph is one where each vertex is adjacent to all the other vertices and a complete graph on n vertices is denoted by K_n. The following result tells us when the graph $G_N(R)$ is complete.

Theorem 2.2. *The nil clean graph $G_N(R)$ is complete if and only if R is nil clean ring.*

Example 2.3. *The current illustration will lead us to the subsequent declaration.*

(a) $G_N(\mathbb{Z}_2)$ (b) $G_N(\mathbb{Z}_3)$ (c) $G_N(\mathbb{Z}_{13})$

Figure 2:

Corollary 2.4. *For any prime number $p \neq 2$, $G_N(\mathbb{Z}_p)$ has two isolated vertices $\bar{1}$ and $\overline{p-1}$ and the other vertices have a degree value of 1. Morovere, the number of edges equals to $\frac{p-3}{2}$.*

The next result specifies the degree of each vertex in graph $G_N(R)$ where $R = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \mathbb{Z}_{p_n}$, for some prime integers $p_1, p_2, \cdots, p_n$.

Proposition 2.5. *Let $(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n) \in R$ where $R = \mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\cdots\mathbb{Z}_{p_n}$ for some prime integers $p_1,p_2,\cdots,p_n$. Then*

(i) $deg(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n) = 2^n - 1$, if $\bar{x}_j \notin U'(R)$ for some $1 \le j \le n$ and $\bar{x}_i \ne \bar{0}$ for all $1 \le i \le n$.

(ii) $deg(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n) = 2^n - 2$, if $\bar{x}_i \in U'(R)$ for all $1 \le i \le n$ and $\bar{x}_i \ne \bar{0}$ for all $1 \le i \le n$.

(iii) $deg(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n) = 2^{n-k}(p_{i_1}p_{i_2}\cdots p_{i_k}) - 1$, if $\bar{x}_{i_j} = \bar{0}$ for some $1 \le j \le k$ and $\bar{x}_{ij} \ne \bar{0}$ for all $j > k$ and $\bar{x}_{ij} \notin U'(R)$ for some $j > k$.

(iv) $deg(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n) = 2^{n-k}(p_{i_1}p_{i_2}\cdots p_{i_k}) - 2$, if $\bar{x}_{i_j} = \bar{0}$ for some $1 \le j \le k$ and $\bar{x}_{i_j} \ne \bar{0}$ and $\bar{x}_{ij} \in U'(R)$ for all $j > k$.

For distinct vertices x and y of G, let $d(x,y)$ be the length of the shortest path from x to y and in case there is no such path, we define $d(x,y) = \infty$. The diameter of G is $diam(G) = \sup\{d(x,y) :$ x and y are distinct vertices of G$\}$. In the event that u and v are two adjacent vertices, then we type in $x \sim y$. The diameter of graph $G_N(R)$ where $R = \mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\cdots\mathbb{Z}_{p_n}$ for some prime integers $p_1,p_2,\cdots,p_n$, is located in the result that follows.

Theorem 2.6. *Let $R = \mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\cdots\mathbb{Z}_{p_n}$ for some prime integers $p_1,p_2,\cdots,p_n$. Then $diam(R) = 4$.*

Proof. Let $a = (\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)$ and $b = (\bar{y}_1,\bar{y}_2,\cdots,\bar{y}_n)$ be two nonzero vertices in R.
Case 1: If for all i, $x_i \ne 0 \ne y_i$ then

$$(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}j\cdot\bar{x}_n) \sim (\bar{0},\bar{0},\cdots,\bar{x_j}^{-1}\cdots\bar{0}) \sim (\bar{y_1}^{-1},\bar{y_2}^{-1},\cdots,\bar{y}_{j-1}^{-1},\bar{0},\bar{y}_{j+1}^{-1},\cdots\bar{y_n}^{-1}) \sim (\bar{y_1},\bar{y_2},\cdots,\bar{y_n})$$

is a path between a and b and so $d(a,b) = 3$.
Case 2: If for some $1 \le i,j \le n$, $\bar{x}_i \ne \bar{0}$, $\bar{y}_j \ne \bar{0}$, $\bar{x_{i_j}} \ne \bar{0}$ and $\bar{y_{i_t}} \ne \bar{0}$, then

(1) If $j \ne t$ then

$$(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_{i_j},\cdots,\bar{x}_n) \sim (\bar{0},\bar{0},\cdots,\bar{x}_{i_j}^{-1}\cdots\bar{0}) \sim (\bar{0},\bar{0},\cdots,\bar{y}_{i_t}^{-1},\cdots\bar{0}) \sim (\bar{y_1},\bar{y_2},\cdots,\bar{y_n})$$

is a path between a and b and so $d(a,b) = 3$.

(2) If $j = t$ then

$$(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_{i_j},\cdots,\bar{x}_n) \sim (\bar{0},\bar{0},\cdots,\bar{x}_{i_j}^{-1}\cdots\bar{0}) \sim (\bar{y}_{i_t},\bar{0},\cdots,\bar{0}) \sim (\bar{0},\bar{0},\cdots,\bar{y}_{i_t}^{-1}\cdots\bar{0}) \sim (\bar{y_1},\bar{y_2},\cdots,\bar{y_n})$$

is a path between a and b and so $d(a,b) = 4$ □

The girth of a graph G is the length of the shortest cycle in G, indicated by $gr(G)$. Note that if there is no cycle in G, at that point $gr(G) = \infty$.

Remark 2.7. *Let $R = \mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\cdots\mathbb{Z}_{p_n}$ for some prime integers $p_1,p_2,\cdots,p_n$. then we have the following cycle:*

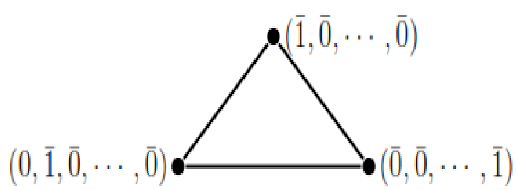

Figure 3: $G_N(\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\cdots\mathbb{Z}_{p_n})$

Theorem 2.8. *Let $R = \mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\cdots\mathbb{Z}_{p_n}$, for some prime integers $p_1,p_2,\cdots,p_n$. Then $N_G(R)$ is a connected graph and $diam(G_N(R)) = 4$. Moreover, $gr(G_N(R)) = 3$.*

Let $R = \mathbb{Z}_{p^n}$ for prime integer p and $n \in \mathbb{N}$. Then $Nil(R) = \{0, p, \cdots, (p^{n-1} - 1)p\}$. such we know $Nil(R_1 \times R_2 \times \cdots \times R_t) = Nil(R_1) \times Nil(R_2) \times \cdots \times Nil(R_t)$ so

$$Nil(\mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \cdots \times \mathbb{Z}_{p_n^{\alpha_n}}) =$$
$$\{(\bar{0}, \bar{0}, ..., \bar{0}), (p_1, \bar{0}, \cdots, \bar{0}), (2p_1, \cdots, \bar{0}), \cdots, ((p_1^{n-1} - 1)p_1, (p_2^{n-1} - 1)p_2, \cdots, (p_n^{n-1} - 1)p_n)\}$$

therefor we have the following result:

Proposition 2.9. *Let $R = \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \cdots \times \mathbb{Z}_{p_n^{\alpha_n}}$ for prime integer $p_i \neq 2$ and $\alpha_i \geq 2$ Then $diam(G_N(R)) = 2$ and $gr(G_N(R)) = 3$.*

Proof. For all $X = (x_1, x_2, \cdots, x_n), Y = (y_1, y_2, \cdots, y_n) \in R$ where $x_i, y_i \in \mathbb{Z}_{pi}$,

1. If $X \in Nil(R)$ or $Y \in Nil(R)$ then $X \sim Y$ is a path between X and Y and $d(x, y) = 1$.
2. If $X, Y \notin Nil(R)$ then $X \sim (p_1, p_2, \cdots, p_n) \sim Y$ is a path between X and Y and $d(X, Y) = 2$ so $diam(G_N(R)) = 2$. □

A clique of a graph G is defined as a complete subgraph of G. The clique number of G, denoted by $\omega(G)$, is the number of vertices in a largest complete subgraph of G. In case $K_n \subseteq G$ for each integers $n \geq 1$, we set $\omega(G) = \infty$. A k-coloring of G is an assignment of k colors $\{1, \cdots, k\}$ to the vertices of G, one color to each vertex, so that adjacent vertices are colored differently. The graph G is k-colorable in the event that G features a proper k-coloring. The chromatic number of G, $\chi(G)$, is the minimum k for which G is k-colorable.

Theorem 2.10. *For any prime number p,*

i. If $p = 2$ or $p = 3$, then $\chi(G_N(\mathbb{Z}_p)) = \omega(G_N(\mathbb{Z}_p)) = 1$

ii. If $p \neq 2, 3$, then $\chi(G_N(\mathbb{Z}_p)) = \omega(G_N(\mathbb{Z}_p)) = 2$.

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Laplacian Eigenvalue Distribution and Graph Parameters

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Abstract

For a graph $G = (V(G), E(G))$ of order n. The *adjacency matrix* of G is denoted by $A(G)$, and the *Laplacian matrix* is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees. The eigenvalues of $L(G)$ are called the *Laplacian eigenvalues* of G. The multiplicity of a Laplacian eigenvalue μ in a graph G is denoted by $m_G(\mu)$, while the number Laplacian eigenvalues of G in an interval I is denoted by $m_G I$. It is well known that $m_G[0,n] = n$ for any graph G, however it is not well understood how the eigenvalues are distributed in the interval $[0,n]$. Many researchers have focused on the bound of $m_G I$ for some subinterval I of $[0,n]$. We show that all graphs $G \neq C_3, C_7$ with minimum degree at least two, $m_G[1,n] \geq \beta'(G) + 1$, where $\beta'(G)$ is the edge covering number of G. We present a short proof of the known result that $m_G(n-1,n] \leq \kappa(G)$, where $\kappa(G)$ is the vertex connectivity of G. Additionally, we classify all trees T such that $m_T(n-i,n] = j$, for $1 \leq i,j \leq 2$. For G with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, we determine the classes of graphs that satisfy the condition $m_G[0,d_1] = 2$, $m_G[d_n,n] = 2$ and $m_G[d_{n-1},n] = 2$.

1 Introduction

Let $G = (V,E)$ be a simple graph. The minimum number of edges needed to cover all vertices is called the *edge covering number* of G and denoted by $\beta'(G)$. The *vertex-connectivity* of G, is the minimum size of a vertex set S such that $G \backslash S$ is disconnected, and demote by $\kappa(G)$.

2 Main results

Theorem 2.1. *Let G be a graph of order n. If G' is the graph obtained from $G \circ K_1$ by adding a vertex u adjacent to some vertices of G, then $m_{G'}[2,n] = \gamma(G \circ K_1) = n$.*

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[*]Speaker

subjclass[2010]: 05C50, 05C69, 15A18
keywords: Laplacian Eigenvalue Distribution, edge covering number, vertex-connectivity

Theorem 2.2. *If G is a graph of order n with no isolated vertex, then*

$$m_G[1,n] \geq \beta'(G),$$

and the equality holds if $\mu(G) = q(G)$.

Theorem 2.3. *There is no tree of order n such that $m_T(n-2,n] = 2$.*

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Representation Theory of Symmetric Groups and Some Combinatorial Results

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Abstract

In this talk, we show a new formulation of the standard character of the symmetric group S_n and we use it to obtain some combinatorial relations.

1 Introduction

Symmetric groups play a fundamental role in algebra, combinations, and representation theory. These groups arise naturally in various mathematical contexts, such as permutation groups and group actions. The study of symmetric groups provides deep insights into the behavior of symmetric structures and has wide-ranging applications in different areas of mathematics. In this paper, we delve into the representation theory of symmetric groups, exploring their irreducible representations and the orthogonal relations between them. Then as an application we obtain some combinatorial relations.

The representation theory of symmetric groups is a branch of mathematics that studies how symmetric groups, which are groups of permutations, can be represented by matrices or linear transformations. Symmetric groups have a rich structure due to their connection with combinatorial objects, such as permutations of a finite set. In this theory, irreducible representations play a crucial role. An irreducible representation of a symmetric group is a representation that cannot be further decomposed into smaller, non-trivial sub-representations. To construct all irreducible representations of symmetric groups, one approach is through the use of Young tableaux. Young tableaux provide a combinatorial tool to describe the irreducible representations of symmetric groups. By associating a Young tableau with a specific shape to each irreducible representation, one can determine the dimension and other properties of the representation. The construction of all irreducible representations involves finding all possible Young tableaux, corresponding to distinct shapes, and determining their associated representations. Another approach to constructing irreducible representations of symmetric groups is through the use of the character theory. The character of a representation is a function that associates each group element with the trace of the

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subjclass[2010]: 20C30, 20B30
keywords: symmetric groups, irreducible representation, fixed points

corresponding matrix or linear transformation. By studying the characters of the irreducible representations, mathematicians can determine the number and dimensions of these representations. For further information about the representation of symmetric groups, we recommend referring to Chapter 8 of [1].

Let S_n be the symmetric group of degree n. S_n acts on $\mathbb{C}^n$ by permuting basis vectors, which defines a representation od degree n. This representation is not irreducible and has a 1-dimensional sub-representation associated with invariant subspace spanned by the vector

$$e_1 + e_2 + \ldots + e_n,$$

representing the trivial representation. A complementary subspace to this is

$$V = \{a_1e_1 + \ldots + a_ne_n \,|\, a_1 + \ldots + a_n = 0\}$$

which often referred to as "the standard representation" of S_n. The standard representation is both irreducible and faithful.

The fix point set of a permutation σ in S_n, denoted by $\mathrm{Fix}_n(\sigma)$, is defined as:

$$\mathrm{Fix}_n(\sigma) = \{i \in \{1, 2, \ldots, n\} \,|\, \sigma(i) = i\}$$

In this definition, $\mathrm{Fix}_n(\sigma)$ represents the set of elements that are fixed by the permutation σ, i.e., the elements that remain unchanged under the permutation.

We call the character associated with the standard representation of S_n as standard character and we denote χ_n. By [[2] Proposition 13.24], the standard character evaluates to

$$\chi_n(\sigma) = |\mathrm{Fix}_n(\sigma)| - 1$$

for permutation σ of S_n.

The standard character provides valuable information about the behavior of permutations in S_n. It tells us the cardinality of the fix point set of a permutation, and by subtracting one, ensures that the identity element of S_n is mapped to 0. The standard character is an important tool in the representation theory of symmetric groups.

2 Standard Representation and Fix Point Set

In this section we give a new formulation of the standard character χ_n, which we will use it to obtain some result in the next section.

Definition 2.1. *Let S_n be the symmetric group of n points. We define subsets $F_{n,i}$ of S_n for $0 \le i \le n$ as follows:*

$$F_{n,i} = \{\pi \in S_n \mid |\mathrm{Fix}_n(\pi)| = i\}$$

Theorem 2.2. *Let S_n be the symmetric group of n points and $X \subseteq S_n$.*

1. $S_n = \bigcup_{i=0}^{n} F_{n,i}$ *(Disjoint Union)*
2. $X = \bigcup_{i=0}^{n} (X \cap F_{n,i})$ *(Disjoint Union)*
3. χ_n *has a constant value of* $i - 1$ *on elements of* $F_{n,i}$.
4. $\sum_{\pi \in X} \chi_n(\pi) = \sum_{i=0}^{n} (i-1)|X \cap F_{n,i}|$.
5. *For* $0 \le k < n$ *and* $\pi \in S_{n-k}$, $|\mathrm{Fix}_{n-k}(\pi)| = |\mathrm{Fix}_n(\pi)| - k$.
6. *For* $0 \le k < n$ *and* $\pi \in S_{n-k}$, $\chi_{n-k}(\pi) = \chi_n(\pi) - k$.

3 Some Combinatorial Results

As an application of section 2, we obtain some combinatorial relations.

Theorem 3.1. *Let S_n be the symmetric group of n points. Then*

1. $\sum_{i=0}^{n}(i^2-2i+1)|F_{n,i}| = n!$.
2. $\sum_{i=0}^{n}(i-1)|F_{n,i}| = 0$.
3. $\sum_{i=0}^{n}(i^2-1)|F_{n,i}| = n!(n-1)$.

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Organized Fraud detection using Graph Theory and Poisson Process

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Abstract

Fraud in the insurance industry is a prevalent issue, particularly in the form of organized schemes involving deliberate accidents and staged scenes. This paper proposes an algorithm designed to achieve three primary objectives. Firstly, accidents are modeled using network graph theory, with the subsequent identification of suspicious fraud clusters through the application of a Poisson random process. Secondly, the algorithm calculates the correlation between individuals involved in suspicious activity using connectivity metrics and the Menger's theorem. It also examines the probability of such accidents occurring by applying local connectivity numbers in the Poisson process. This process enables the validation of each accident and individual through the assignment of a label. Lastly, while most research algorithms in fraud detection utilize data mining or artificial intelligence, this paper overcomes the challenges posed by highly unbalanced data, including overfitting and reduced accuracy.

1 Introduction

Insurance fraud is a deceptive practice aimed at defrauding insurance companies for financial gain. It has been a persistent issue since the inception of commercial enterprises, resulting in substantial financial losses for insurance companies on an annual basis. This type of fraud manifests in various forms across all sectors of insurance, including a wide range of exaggerated claims, deliberate accidents, and damages. Organized automobile insurance fraud, which is the focus of this paper, is often performed by groups in a structured manner, leading to higher costs for insurance companies and subsequently increased premiums for policyholders. Despite advancements in fraud detection, the financial impact of these scams on insurance companies continues to escalate.

In the past, the level of organized insurance fraud that could have significant financial consequences for the insurer was not enough to justify the exploration of potential solutions and the allocation of resources towards addressing it. However, there has been a notable shift in this scenario, prompting insurance companies to consider thoroughly investigating the elements influencing insurance fraud. Today, with

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the need to detect fraud in various areas, the use of data mining techniques and machine learning, such as artificial neural networks, fuzzy logic, and genetic algorithms, have become common tools for fraud detection due to their high ability in modeling complex problems.

Graph theory is an alternative method that can be employed to identify organized fraudulent activities. Initially, the issue is mathematically represented by modeling the network of accidents as a graph, thereby enabling the initial detection of organized fraud. Subsequently, computer science concepts are utilized to accurately identify the suspicious fraud network. In structures such as national incident databases, an extensive volume of data is available, making the task of finding relationships between them challenging and sometimes unfeasible. To address this issue, data mining and machine learning techniques are used. However, they have significant shortcomings, including computational complexity when handling large datasets and the existence of imbalanced datasets. While these techniques employ methods such as over-sampling or under-sampling to manage imbalanced data, they encounter additional challenges such as overfitting and reduced accuracy. Moreover, despite the discussion of self-validation of incidents in [1] and [2], no validation has been done for individual cases. Therefore, in this paper, we try to address these shortcomings through the application of mathematical models, simultaneously investigating the probability of occurrence of suspicious events and introducing an algorithm for it.

2 Main results

In this section, we examine the main results of the paper. To this end, we first demonstrate that car accidents are a random process.

Theorem 2.1. *The random network between cars over a specific period of time and geographical area is a Poisson process.*

We then use graph theory to model accidents between cars. In this model, the cost that the insurer pays to the policyholders for an accident is included using Chebyshev's inequality. Clearly, the existence of some regular structures in this network contradicts the randomness of car collisions; therefore, they can be considered as suspicious cases. A novel algorithm with a polynomial time complexity is introduced to identify such structures in the accident network (first step of the algorithm). We then deal with the probability of each accident occurring in the network. In other words, we demonstrate that:

Theorem 2.2. *Suppose N is a random network. If X represents the number of distinct paths between any two arbitrary vertices u and v, then X is a Poisson random variable with parameters (P, p) where P represents the number of paths and p is the probability of the occurrence of a random chain between u and v.*

Now, by utilizing connectivity matrices in the graph, the Menger theorem, and the parameters of the Poisson process, we initially assign a label based on the probability of each accident occurring and subsequently employ these labels to validate each insured person (second step of the algorithm).

This labeling can consider multiple policies to facilitate or penalize the insurance process in addition to the current floatability of insurance. Moreover, the insurer can provide a balanced point between the insurer and the policyholders so that both parties can achieve their interests. The second suggestion is to create and examine a broader network that includes all stakeholders in an organized fraud. More precisely, the network allocates labels to the main stakeholders who benefit from an accident, depending on their profit, to be examined. Examining this issue enables the insurer to adopt different policies for dealing with different stakeholders in an accident, such as the insured, car occupants, repairers, etc., to reduce financial loss and improve public trust.

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Complete forcing number of tori and hypercubes

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Abstract

A forcing set for a perfect matching in a graph G is defined as a subset of the edges of that perfect matching such that there exists a unique perfect matching containing them. A complete forcing set for a graph is a subset of its edges, such that it intersects the edges of every perfect matching in a forcing set of it. The size of a smallest complete forcing set of a graph is called the complete forcing number. In this paper, we find upper and lower bounds of for the complete forcing number of hypercube graphs.

1 Introduction

For the basic definitions and notation of graph theory, we follow the reference [9]. Let $G = (V, E)$ be a simple graph. On the vertex set $V(G)$ and the edge set $E(G)$, two edges are called incident if they share an endpoint vertex. Two edges that are not incident are called disjoint. A set of pairwise disjoint edges is called a matching of G. A perfect matching of G is a matching that covers all vertices. Let M be a perfect matching for G. A subset $S \subseteq V(G)$ is called a forcing set of M if M is the unique perfect matching containing it. The size of the smallest forcing set of M is called its forcing number. The maximum forcing number of G is the maximum among the forcing numbers of all perfect matchings of G and is denoted by $F(G)$. The minimum forcing number of G denoted by $f(G)$, is defined in a similar way.

Xu et al. [1] proposed the concept of the complete forcing set of G, which is defined as a subset of $E(G)$ on which the restriction of each perfect matching M is a forcing set of M. The smallest cardinality of any complete forcing sets is called the *complete forcing number* of G, and is denoted by $cf(G)$.

The main application of this parameter is in studying the Kekule structures of benzoid graphs in Chemistry.

Let S be a nonempty proper subset of $V(G)$. The set of all edges of G having exactly one end-vertex in S is denoted by $\delta_G(S)$ (or simply $\delta(S)$) and is called an edge cut of G. A bridge of G is an edge cut of G consisting of exactly one edge.

A set $S \subseteq V(G)$ is called a 2-independence set of G if, for every two distinct vertices $u, v \in S$, $d(u, v) > 2$, where $d(u, v)$ is the distance between u and v. An automorphism of a graph G, is a one-to-one mapping from the vertex set of G to itself that preserves the adjacency of the vertices. The set of automorphisms of

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G is known to be a group which naturally acts on the set of the vertices as well as the edges. If the action of this group on the set of vertices (edges) is transitive, then the graph is called vertex (edge) transitive.

The Cartesian product $G \times H$ of two graphs G and H is a graph with the vertex set $V(G) \times V(H)$, such that two vertices (u, v) and (u', v') are adjacent, if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. A hypercube graph, denoted as Q_n, is constructed as the Cartesian product of n complete graphs of order 2 (i.e., K_2). Its equivalent representation involves assigning a binary string to each vertex, and two vertices are adjacent if their binary strings differ in exactly one bit.

It is known that the Cartesian product of vertex-transitive graphs is also vertex-transitive. This does not hold for the case of edge-transitive graphs, though. For instance, the Cartesian product of two cycles of different lengths is not edge-transitive. Therefore, Q_n is both vertex and edge-transitive graph for every value of n.

Let P_n and C_n be the path and the cycle with n vertices, respectively. Chang et al. [3] obtained the formula for the complete forcing number of rectangular polyominoes ($P_m \times P_n$). The obvious next step can be studying the problem for cylinders (i.e. $P_m \times C_n$) and tori (i.e $C_m \times C_n$) graphs.

Recently, He and Zhang [2] established that the complete forcing number of a graph is no more than twice its cyclomatic number and presented a method for constructing a complete forcing set for a graph. Using this method, they provided the formula for the complete forcing number of wheels (W_n) and cylinders.

1.1 Our Contribution

In this work, motivated by the results of [1], we consider the problem of finding the complete forcing number of the graphs $C_m \times C_n$ for even $m.n$, Note that when both m, n are odd, the product graph has no perfect matching and hence the problem is trivial in that case. We show that when both m, n are even numbers, then the complete forcing number belongs to the set $\{m.n, m.n+1, \cdots, m.n+4\}$. When n is even and m is odd, the possibilities reduces to one of the cases $\{m.n + \frac{n}{2}, m.n + \frac{n}{2} + 1, m.n + \frac{n}{2} + 2\}$. The case odd n and even m is treated by symmetry. We formally state this result as follows:

Theorem 1.1. *Suppose that $n, m \geq 3$.*

(i) For odd m and even n:

$$mn + \frac{n}{2} \leq cf(C_m \times C_n) \leq mn + \frac{n}{2} + 2$$

(ii) For even m and odd n :

$$mn + \frac{m}{2} \leq cf(C_m \times C_n) \leq mn + \frac{m}{2} + 2$$

(iii) For both even m and n:

$$mn \leq cf(C_m \times C_n) \leq mn + 4$$

We also propose a general framework for finding a lower bound for the complete forcing number of an edge-transitive graph. In fact, finding the complete forcing number of Q_n is closely related to finding large-size binary error correction codes of minimum distance 3. There from, we obtain an upper bound for the complete forcing number of Q_n that almost matches our lower bound. The main result of this part is as follows. For every positive real number $\alpha < 1$, if n is large enough, then the complete forcing number of Q_n is larger than α fraction of all the edges. The following statements contains the formal statement of this result.

Theorem 1.2. *For every constant $c < 1$, there exists a value n such that*

$$cf(Q_n) \geq c.|E(Q_n)| = c.n.2^{n-1}$$

Theorem 1.3. *For every $n \geq 1$, We have*

$$cf(Q_n) \leq n\left(2^{n-1} - 2^{n-\lceil \log (n+1) \rceil}\right)$$

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On the Scale-embedding of weighted graphs into hypercubes

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Abstract

In this paper, we study the problem of scale embedding of simple weighted graphs into hypercubes. In the first step, we convert the problem into a question about the existence of a non-negative solution to a system of linear equations and determine whether there are any scale embeddings or not. Further, a special case of the linear equation is also investigated and the existence of a non-negative solution is verified in this case.

1 Introduction

1.1 Preliminaries and notations

Let the graph $G = (V, E)$ be a simple connected graph and $d : V(G) \times V(G) \to \mathbf{Z}_{\geq 0}$ the distance function as the length of the shortest path between each pair of vertices. The hypercube graph Q_n has a vertex set consisting of all binary strings with length n, and two vertices adjacent when their strings differ by a single bit. l_1-distance is defined by $d_{l_1}(x, y) = \sum_{i=1}^{m} |x_i - y_i|$ for each $x, y \in \mathbf{R}^m$ and $\left(\mathbf{R}^m, d_{l_1}\right)$ called l_1-space for some integer $m \geq 1$.

1.2 Prior works

Definition 1.1. *A "λ-embedding" of graph G into the hypercube Q_n is any mapping $\phi : V(G) \to V(Q_n)$ such that for all vertices $x, y \in V(G)$ we have $d(\phi(x), \phi(y)) = \lambda d(x, y)$.*

If there exist positive integers λ and n for a graph G, then the mapping is called scale embedding. Also, graph G is λ-embedding into a hypercube. 1-embedding graphs are called "isometric embedding" too. Embeddings graphs into other graphs can help analyze graph distances more easily. A special kind of embeddings are those into Hamming graphs, which are Cartesian products of complete graphs. A special case of Hamming graphs is a hypercube. Hamming embedding of graphs has many applications in

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different fields. For instance, they can model the variations of DNA sequences in molecular engineering [4]. They can also enable optimal information transfer in communication theory without knowing the whole network structure [6]. In linguistics, they can measure the similarity of linguistic objects using simple predicates as vectors [5]. In coding theory, they can optimize the error-detection of codes based on Hamming distance [7].

Definition 1.2. *Graph G is l_1 if it can be embedded into a l_1-space.*

It is shown that a graph is l_1 if and only if it is scale embeddable into a hypercube. (See [1]) In [2], Shpectorov showed that any l_1-graph is an isometric subgraph of a Cartesian product of cocktail party graphs and halved cubes. The property of G being an l_1-graph can be recognized in a polynomial time [2].

2 Main results

We aim to investigate the scale embedding of weighted graphs in this section. We convert it into linear equations, a general case, and a special case. We study the λ-scale embedding to weighted connected graphs. Here, the shortest path between a pair of vertices will be concerning weights. Isometric Hamming embedding of weighted graphs has been studied recently in [8]. The main result of this paper states that: a weight-minimal weighted graph has a Hamming embedding if and only if a Hamming embedding exists for each factor of its canonical isometric representation. In contrast to the unweighted case, determining if an arbitrary weighted graph permits a hypercube embedding is NP-hard. [3]

2.1 Problem setup

λ-scale embedding for a weighted graph can be converted as a linear equation where w is the weight vector on edges as $w = \left(w_1, w_2, \ldots, w_{\binom{n}{2}}\right)$.
$Cx = w$ where C is a $\binom{n}{2} \times 2^n$ matrix and x is a vector $2^n \times 1$. When we define C as:
For each $i \neq j$, and set $s \subset \{1, 2, \ldots, n\}$, the array in the intersection of row $\{i, j\}$ and column s is 1 if and only if $|\{i, j\} \cap s| = 1$.

Theorem 2.1. *A weighted graph G with weight vector w is λ-scale embeddable if the equation $Cx = w$ has a non-negative rational solution.*

A special case can be just using binary codes with 2 ones instead of using all binary strings. In this case, instead of matrix C, we will have matrix L as we find it later.
Consider martix A which is $\binom{n}{2} \times n$ for a weighted graph G with n vertices as below:
Each row of A has exactly 2 ones related to different pairs of vertices such as i, j among all n vertices. Matrix A multiplication to its transpose is a $\binom{n}{2} \times \binom{n}{2}$ matrix and equals to $L + 2I$. we have the following equities:(where I is the Identity matrix and J is all 1 matrix)

$$L = AA^T - 2I, \quad (n-2)I + J = A^T A \tag{1}$$

We are interested in the w's that the $Lx = w$ has a non-negative solution.

Theorem 2.2. *A weighted graph G is λ-scale embeddable by binary codes with 2 ones if the equation $Lx = w$ has a non-negative solution.*

Example 2.3. *Consider simple graph $G = K_3$, it is easy to see G is not a 1-scale embeddable into any hypercube but, it is 2-scale embeddable and it can be seen in the below:*

$$\left(\left(\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix}\right) - \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow x_1 = x_2 = x_3 = \frac{1}{2}$$

We see that the equation $Cx = w$, has the non-negative rational solution of $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then the minimum scale can be the common denominator of rational entries of the solution; here it will be 2. Moreover, the binary codes on vertices are 110, 101, 011.

In the next example, we will see an example of a weighted graph that is not λ-scale embeddable with binary codes with 2 ones.

Example 2.4. *Consider the weighted K_5 with weights of $\{w_{12} = 1, w_{13} = 3, w_{14} = 1, w_{15} = 3, w_{23} = 2, w_{24} = 1, w_{25} = 2, w_{34} = 3, w_{35} = 2, w_{45} = 3\}$. The equation of $Lx = w$ for this graph has a solution with negative entries;* $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, 0, \frac{1}{2}, 2, \frac{1}{2})$.

Lemma 2.5 (Farkas)**.** *Let $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$. Then exactly one of the following two assertions is true:*

- *There exists an $x \in \mathbf{R}^n$ such that $Ax = b$ and $x \geq 0$.*
- *There exists an $y \in \mathbf{R}^m$ such that $A^T y \geq 0$ and $b^T y < 0$.*

Lemma 2.6. *The inverse of matrix L as follows: (all of matrices L, I, J, L^{-1} are $\binom{n}{2} \times \binom{n}{2}$)*

$$L^{-1} = \tfrac{1}{2n-8} L + \tfrac{6-n}{2n-8} I - \tfrac{1}{(n-2)(n-4)} J$$

Proof. First, we find the $(L + 2I)^2$:

$$(L + 2I)^2 = AA^T AA^T = A\big((n-2)I + J\big)A^T = (n-2)(L + 2I) + 4J = (n-2)L + 2(n-2)I + 4J$$

Also, we know that $LJ = 2(n-2)J$ which implies that $J = \frac{1}{(2n-4)} LJ$. By simplification, we get L^{-1}. $\square$

Remark 2.7. *We also showed that L^{-1} exists by finding its eigenvalues which all are non-zero. For this, we used the fact that $AA^T, A^T A$ have the same non-zero eigenvalues.*

More precisely, we have:
Matrix I has the eigenvalue 1 with multiplicity of it order, here $\binom{n}{2}$. Matrix J has an eigenvalue of $\binom{n}{2}$ with a multiplicity of one and its remained eigenvalues are 0. As we have 1, matrix L has an eigenvalue of $n - 2 + \binom{n}{2} - 2$ once and the other ones are $n - 4$. These eigenvalues are non-zero for $n > 4$ and matrix L^{-1} does exist.

Corollary 2.8. *When all of the weights of the graph are constant and equal to w, the λ-scale embedding exists.*

Proof. We can easily see that the $L^{-1}w$ is a non-negative vector. In fact, it is equal to $\frac{n-4}{2(n-2)(n-4)}$ which is non-negative for all $n \geq 4$. $\square$

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Degree distance and Gutman index of a graph

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Abstract

The degree distance was introduced by Dobrynin, Kochetova and Gutman as a weighted version of the Wiener index. In this paper, we investigate the degree distance and Gutman index of some graphs by using the adjacency and distance matrices of a graph.

1 Introduction

For a graph G, let $V(G)$, $E(G)$ and $\overline{G}$ denote the set of vertices, the set of edges and the complement of G, respectively. If G is a connected graph and $u, v \in V(G)$, then the *distance* $d(u,v)$ between u and v is the length of a shortest path connecting u and v.

Furthermore, the *diameter* $diam(G)$ of G is defined by $diam(G) = \max\{d(u,v) \,|\, u, v \in V(G)\}$.

Let G be a finite, simple, connected, undirected graph with p vertices and q edges. In what follows, we say that G is an (p,q)*-graph*. The *adjacency matrix* of G is the $p \times p$ matrix $A = A(G)$ whose (i,j) entry, denoted by a_{ij}, is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = [a_{ij}]_{m\times n}$. Then, we define

$$S(A) = \sum_{1\le i\le m,\ 1\le j\le n} a_{ij}.$$

For $k = 1, 2, \cdots, \alpha$ where α denotes the diameter of graph $G(p,q)$, we define

$$A_k = [a_{ij}^k]_{p\times p},$$

where $a_{ij}^k = \begin{cases} 1 & d(v_i, v_j) = k \\ 0 & \text{otherwise.} \end{cases}$ The *distance matrix* of G is the $p\times p$ matrix D_G whose (i,j) entry, denoted by d_{ij}, is defined by

$$d_{ij} = \begin{cases} d(v_i, v_j) & \text{if } v_i \ne v_j \\ 0 & \text{otherwise.} \end{cases}$$

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keywords: Degree distance, adjacency matrix, distance matrix

The oldest and most studied degree-based structure descriptors are the *first and second Zagreb indices* [6], defined as

$$M_1(G) = \sum_{v\in V(G)} (d_G(v))^2 \qquad \text{and} \qquad M_2(G) = \sum_{uv\in E(G)} (d_G(u))(d_G(v)) \ .$$

It has been shown that the first Zagreb index obeys the identity [4]

$$M_1(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v)) \ .$$

The first investigation of the sum of distance between all pairs of vertices of a (connected) graph was done by Harold Wiener in 1947, who realized that there exists a correlation between the boiling points of paraffins and this sum [7]. Eventually, the distance–based graph invariant,

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v) = \frac{1}{2} \sum_{u,v\subseteq V(G)} d(u,v)$$

.

The degree distance was introduced by Dobrynin and Kochetova [3] and Gutman [5] as a weighted version of the Wiener index. The *degree distance* $DD(G)$ of a graph G is defined as

$$DD(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v)[d_G(u) + d_G(v)] = \frac{1}{2} \sum_{u,v\in V(G)} d(u,v)[d_G(u) + d_G(v)]$$

with the summation runs over all pairs of vertices of G. The degree distance is also known as the Schultz index in chemical literature; In [5], Gutman showed that if G is a tree on n vertices, then $DD(G) = 4W(G) - n(n-1)$. In [2], *Gutman index* $Gut(G)$ of a graph G is defined as

$$Gut(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u)d_G(v)d(u,v).$$

For more details on Gutman index, we refer to [1].

2 Main Result

In this section, we give our main results and their proofs:

Lemma 2.1. *Let D_G be the distance matrix of graph G. Then, $S(D_G) = 2W(G)$. In particular, if $diam(G) = 2$ then $W(G) = 2\binom{p}{2} - q$.*

Proof.

$$\begin{aligned} S(D_G) &= \sum_{i=1}^{p}\sum_{j=1}^{p} d(v_i,v_j) = 2 \sum_{\{v_i,v_j\}\subseteq V} d(v_i,v_j) \\ &= 2W(G). \end{aligned}$$

In particular, we have

$$\begin{aligned} 2W(G) &= S(D_G) = S(A + 2\overline{A}) \\ &= 2S(A + \overline{A}) - S(A) = 2S(K) - S(A) \\ &= 2p(p-1) - 2q = 4\binom{p}{2} - 2q. \end{aligned}$$

Therefore, $W(G) = 2\binom{p}{2} - q$. □

Lemma 2.2. *Let $G(p,q)$ be a graph, then*

(1) $S(AD_G) = DD(G)$;

(2) *if* $diam(G) = 2$ *then* $DD(G) = 4(p-1)q - M_1(G)$;

(3) *if* $diam(G) = 3$ *and* G *has no triangles, then*

$$DD(G) = 6q(p-1) - 2M_1(G) - N_1(G),$$

and

$$W(G) = \frac{3}{2}p(p-1) - 2q - q'.$$

Lemma 2.3. *et* G *be a* (p,q)*-graph then*

(1) $Gut(G) = \frac{1}{2}S(AD_G A)$.

(2) *If* $diam(G) = 2$, *then* $Gut(G) = 4q^2 - M_1(G) - M_2(G)$.

(3) *If* $diam(G) = 3$ *and* G *has no cycles of size* 3 *then*

$$Gut(G) = 6q^2 - \frac{3}{2}M_1(G) - 2M_2(G) - N_2(G).$$

Theorem 2.4. *Let* $G(p,q)$ *be a tree graph such that* $diam(G) = 4$, *then*

(1) $W(G) = 2(p-1)^2 - M_2(G)$.

(2) $DD(G) = (p-1)(7p-8) - 4M_2(G)$.

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On characteristic and Laplacian characteristic polynomials of caterpillars

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Abstract

In this paper, we state some results about characteristic and Laplacian characteristic polynomials of a caterpillar $T(m, n, p)$.

1 Introduction

All graphs in this paper are finite and undirected with no loops or multiple edges. Let G be a graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, where $D(G) = diag(d(v_1), \ldots, d(v_n))$ is a diagonal matrix and $d(v)$ denotes the degree of the vertex v in G and $A(G)$ is the adjacency matrix of G. The *characteristic polynomial* and *Laplacian characteristic polynomial* of G are denoted by $\psi_G(\lambda) = det(\lambda I - A)$ and $\varphi(L(G)) = det(\mu I - L(G))$, respectively. Also, denoting eigenvalues and Laplacian eigenvalues of G by $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ and $\mu_1(G) \geq \cdots \geq \mu_n(G) = 0$, respectively. In this talk, we shall use the notation $\lambda_k(G)$ ($\mu_k(G)$) to denote the k^{th} eigenvalue (Laplacian eigenvalue) of G. Two non-isomorphic graphs are said to be *co-spectral* if they have the same eigenvalues with the same multiplicities. The path and the star graph of order n are denoted by P_n and S_n, respectively. A *one-edge connection* of two graphs G_1 and G_2 is a graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{e = uv\}$, where $u \in V(G_1)$ and $v \in V(G_2)$ and is denoted by $G = G_1 \odot_{uv} G_2$.

A *caterpillar* is a tree of order $n \geq 5$ (notice that a tree of order less than 5 is a path or a star) such that removing all the pendant vertices produces a path with at least two vertices. In particular, the caterpillar $T(n_1, \cdots, n_r)$ is obtained from a path P_r and attaching the central vertex of the star $S_{n_i+1} (1 \leq i \leq r)$ to the i^{th} vertex of the path P_r. In the other words,

$$T(n_1, \ldots, n_r) \cong S_{n_1+1} \odot_{u_1 u_2} S_{n_2+1} \odot_{u_2 u_3} \cdots \odot_{u_{r-1} u_r} S_{n_r+1},$$

where u_i is the central vertex of the star S_{n_i+1} for each $(i = 1, \ldots, r)$.

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*Speaker

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keywords: Caterpillar, Characteristic polynomial, Eigenvalue, Laplacian characteristic polynomial, Laplacian eigenvalue.

2 Main results

Let m_i and n_i ($i = 1, 2, 3$) be positive integers. As a first result in this talk, we investigate if $G_1 = T(m_1, m_2, m_3)$ and $G_2 = T(n_1, n_2, n_3)$ are co-spectral, then they are isomorphic.

Let $G = G_1 \odot_{uv} G_2$ and $G'_1(G'_2)$ be the induced subgraph of $G_1(G_2)$ obtained by deleting the vertex $u(v)$ from $G_1(G_2)$. The following result was proved in [2].

Theorem 2.1. [2, Theorem 2.2] $\psi_G(\lambda) = \psi_{G_1}(\lambda)\psi_{G_2}(\lambda) - \psi_{G'_1}(\lambda)\psi_{G'_2}(\lambda)$.

It is well-known that $\psi_{S_{n+1}}(\lambda) = \lambda^{n-1}(\lambda^2 - n)$. By considering this fact and Theorem 2.1, for $r = 2$ we have:

$$\begin{aligned}
\psi_{T(m,n)}(\lambda) &= \psi_{S_{m+1}}(\lambda)\psi_{S_{n+1}}(\lambda) - \lambda^{m+n} \\
&= \lambda^{m-1}(\lambda^2 - m)\lambda^{n-1}(\lambda^2 - n) - \lambda^{m+n} \\
&= \lambda^{m+n-2}[\lambda^4 - (m+n+1)\lambda^2 + mn],
\end{aligned}$$

and so we obtain the following theorem.

Theorem 2.2. *A caterpillar $T(m,n)$ is determined by its spectrum.*

Now, Let $r = 3$ and $G = T(m,n,p)$. Continuing the above method, we have:

$$\begin{aligned}
\psi_G(\lambda) &= \psi_{T_{m,n}}(\lambda)\psi_{S_{p+1}}(\lambda) - \psi_{S_{m+1}}(\lambda)\lambda^n\lambda^p \\
&= \lambda^{m+n-2}[\lambda^4 - (m+n+1)\lambda^2 + mn]\lambda^{p-1}(\lambda^2 - p) - \lambda^{m-1}(\lambda^2 - m) - \lambda^n\lambda^p \\
&= \lambda^{m+n+p-3}[\lambda^6 - (m+n+p+1)\lambda^4 + (mn+mp+np+p)\lambda^2 - mnp] - \lambda^{m+n+p-1}(\lambda^2 - m) \\
&= \lambda^{m+n+p-3}[\lambda^6 - (m+n+p+2)\lambda^4 + (mn+mp+np+m+p)\lambda^2 - mnp],
\end{aligned}$$

and so we obtain the following theorem.

Theorem 2.3. *A caterpillar graph $G = T(m,n,p)$ characterize by by its spectrum.*

Now, as a second result, we prove that any caterpillar $T(m,n,p)$ is determined by its Laplacian eigenvalues. In [1], the Laplacian characteristic polynomial S_{n+1} is equal

$$\varphi(L(S_{n+1})) = \mu(\mu - n - 1)(\mu - 1)^{n-1}.$$

For, any $v \in V(G)$, let $L_v(G)$ be the principal submatrix of $L(G)$ formed by deleting the row and column corresponding to vertex v. The following result was proved in [3].

Theorem 2.4. [3, Lemma 8] *Let $G = G_1 \odot_{uv} G_2$ be the graph, then*

$$\varphi(L(G)) = \varphi(L(G_1))\varphi(L(G_2)) - \varphi(L(G_1))\varphi(L_v(G_2)) - \varphi(L(G_2))\varphi(L_u(G_1)).$$

Now, by using Theorem 2.4 for a caterpillar $G = T(m,n)$, we have:

$$\begin{aligned}
\varphi(L(G)) &= \varphi(L(S_{m+1}))\varphi(L(S_{n+1})) - \varphi(L(S_{m+1}))(\mu - 1)^n - \varphi(L(S_{n+1}))(\mu - 1)^m \\
&= \mu(\mu - m - 1)(\mu - 1)^{m-1}\mu(\mu - n - 1)(\mu - 1)^{n-1} \\
&\quad -\mu(\mu - m - 1)(\mu - 1)^{m-1}(\mu - 1)^n - \mu(\mu - n - 1)(\mu - 1)^{n-1}(\mu - 1)^m
\end{aligned}$$

$$= \mu(\mu - 1)^{m+n-2}[\mu^3 - (m + n + 4)\mu^2 + (mn + 2m + 2n + 5)\mu - (m + n + 2)],$$

and so we obtain the following theorem.

Theorem 2.5. *A caterpillar $T(m, n)$ is determined by its spectrum.*

Finally, By using Theorems 2.4 and 2.5, the Laplacian characteristic polynomial of $G = T(m, n, p)$ is equal to

$$\begin{aligned}\varphi(L(G)) = \varphi(L(T(m,n)))\varphi(L(S_{p+1})) - \varphi(L(T(m,n)))(\mu-1)^p - \varphi(L(S_{m+1}))(\mu-1)^{n+p} \\ = \mu(\mu-1)^{m+n+p-3}[\mu^5 - (m+n+p+7)\mu^4 \\ +(mn+mp+np+5m+4n+5p+18)\mu^3 \\ -(mnp+2mn+3mp+2np+8m+6n+8p+22)\mu^2 \\ +(mn+3mp+np+5m+4n+5p+13)\mu - m - n - p - 3\end{aligned}$$

and so we have the following:

Theorem 2.6. *A caterpillar $T(m, n, p)$ is determined by its spectrum.*

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Characterization of graphs based on chromatic symmetric function

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Abstract

Let G be a graph. The Chromatic symmetric function X_G introduced by Stanley, is a sum of monomial symmetric functions corresponding to proper colorings of G. Using a polynomial with two variables it was proved that the degree sequence of every tree is determined by chromatic symmetric function. In this talk, we provide a simple and short proof for this result. We also show that the degree sequence of $L(G)$ can be deduced from X_G provided that G is a tree, where $L(G)$ is the line graph of G. Moreover, we prove that if G is triangle-free, then $X_{\overline{G}}$ can be derived from X_G.

Joint Work with: S. Akbari, A. Alavi, B. Samimi, E. Zahiri

1 Introduction

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. For $v \in V(G)$, $d(v)$ denotes the degree of v. The *complement* $\overline{G}$ of G is the graph with the vertex set $V(G)$ and two vertices are adjacent in $\overline{G}$ if they are not adjacent in G. A *matching* M in G is a set of pairwise non-adjacent edges and we denote the size of maximum matching of G by $\alpha'(G)$. We call G *triangle-free* if it has no cycle of length 3 as a subgraph. Let P denote the positive integers. A *proper vertex coloring* of G is a function $\kappa : V(G) \to P$ such that $\kappa(v) \neq \kappa(w)$ whenever the vertices v, w are adjacent. Stanley in [3] defined the *chromatic symmetric function* of G as

$$X_G = X_G(x_1, x_2, \dots) = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)},$$

the sum over all colorings κ, where $x_1, x_2, \dots$ are countably infinitely many commuting indeterminates. This definition is invariant under permutations of x_i, so X_G is a symmetric function, homogeneous of degree $n = |V(G)|$. It was proved that the degree sequences of a tree T can be recovered from its chromatic symmetric function see [2]. Here, we provide a simple and short proof for this result.
The *line graph* of G, denoted as $L(G)$, is the graph whose vertex set is $E(G)$; two vertices of $L(G)$ are adjacent if the corresponding edges of G are incident. We show that the degree sequence of $L(G)$ can

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be deduced from X_G provided that G is a tree. Moreover, we prove that if G is triangle-free, $X_{\overline{G}}$ can be derived from X_G. We call the sequence $s_1, s_2, \ldots, s_{n-1}$ the *star sequence* of G if and only if, for every $i = 2, 3, \ldots, n-1$, s_i is equal to the number of subgraphs of G that are isomorphic to $K_{1,i}$.

2 Main results

Theorem 2.1. *Let T be a tree. Then the degree sequence of T can be determined by X_T.*

Theorem 2.2. *Let G be a simple graph of order n with no isolated vertex. It is possible to obtain the degree sequence of G by having its star sequence.*

Theorem 2.3. *Suppose G is a triangle-free graph of order n. Then, the following holds:*

$$\chi(\overline{G}) = n - \alpha'(G)$$

Theorem 2.4. *Suppose G is a triangle-free graph of order n. In that case, $X_{\overline{G}}$ is computable.*

Conjecture 2.5. *The degree sequence of every bipartite graph can be determined by its chromatic symmetric function.*

Conjecture 2.6. *Every triangle-free graph has a unique chromatic symmetric function.*

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A generalization of 3-rainbow domination in graphs

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Abstract

Assume set of 3 colors and to each vertex of a graph G we assign an arbitrary of these colors. If we require that each vertex to set is assigned has in its closed neighborhood all 3 colors, then this is called the generalized 3-rainbow dominating function of a graph G. The corresponding γ_{g3r}, which is the minimum sum of numbers of assigned colors over all vertices of G, is called the g3-rainbow domination number of G. In this paper we introduce this new concept and we present a linear algorithm for determining a minimum generalized 3-rainbow dominating set of P_n, $K_{m,n}$ and $GP(n,3)$.

1 Introduction

Domination and its variations in graphs have been extensively studied, c.[1]. For a graph $G = (V, E)$, a set S is a domination set if every vertex in $V\backslash S$ is adjacent to a vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. We call a dominating set of cardinality $\gamma(G)$ is a $\gamma(G)$-set.
Let G be a graph and let f be a function that assigns to each vertex a set of colors chosen from the set $\{1,2,3\}$; that is, $f : V(G) \to P\{(1,2,3)\}$. If for each vertex $v \in V(G)$ such that $f(v) = \varnothing$ we have $\bigcup_{u\in V(G)} f(u) = \{1,2,3\}$, then f is called 3-rainbow domination function of G. The corresponding γ_{3r}, which is the minimum sum of numbers of assigned colors over all vertices of G, is called the 3-rainbow domination number of G.

Definition 1.1. *Let G be a graph and let f be a function that assigns to each vertex a set of colors chosen from the set $\{1,2,3\}$; that is, $f : V(G) \to P(\{1,2,3\})$, if for each vertex $v \in V(G)$ we have $\bigcup_{u\in N[v]} f(u) = \{1,2,3\}$, then f is called generalized of 3-rainbow dominating function (g3rdf) of G. The weight, $\omega(f)$, of a function f is defined as $\omega(f) = \sum_{v\in V(G)}|f(v)|$. For a graph G, the minimum weight of a g3rdf is called the generalized 3-rainbow dominating number of G, which we denote by a $\gamma_{g3r}(G)$.*

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2 Main results

In this section, we state some of new results on the generalized 3-rainbow domination in graphs.

Lemma 2.1. *we have,*

(a) $\gamma_{g3r}(k_n) = 3$

(b) $\gamma_{g3r}(k_{m,n}) = 6$

(c) $\gamma_{g3r}(k_{1,n}) = 3$

(d) *If* $i \in \{2,3\}$, *then* $\gamma_{g3r}(p_i) = 3$.

(e) *If* $i, n \in N$ *and* $3n+1 \le i \le 3(n+1)$ *then* $\gamma_{g3r}(p_i) = 3n+3$.

Definition 2.2. *Let* $n \ge 3$ *and* k *be relatively prime natural numbers and* $k < n$. *The generalized Petersen graph* $GP(n,k)$ *is defined as follows. Let* C_n, C'_n *be two disjoint cycles of length* n. *Let the vertices of* C_n *be* $u_1, ..., u_n$ *and edges* u_iu_{i+1} *for* $i = 1, ..., n-1$ *and* u_nu_1. *Let the vertices of* C'_n *be* $v_1, ..., v_n$ *and edges* v_iv_{i+k} *for* $i = 1, ..., n$, *the sum* $i+k$ *being taken modulo* n *(throughout this section). The graph* $GP(n,k)$ *is obtained from the union of* C_n *and* C'_n *by adding the edges* u_iv_i *for* $i = 1, ..., n$. *Its obvious that* $GP(n,k) = GP(n, n-k)$. *The graph* $GP(5,2)$ *or* $GP(5,3)$ *is the well-known Petersen graph.*

Theorem 2.3. *For graphs GP(n,3) that* $n \ge 5$ *and* n *and* 3 *be relatively prime numbers,*

$$\gamma_{g3r} \le \begin{cases} 6[\frac{n}{3}] + 3, & \text{if } n \equiv 1(mod3); \\ 6\lceil\frac{n}{3}\rceil, & \text{if } n \equiv 2(mod3); \end{cases} \tag{1}$$

Proof. We use the following partition of V(GP(n,3)):
If $n \equiv 1, (mod3)$, we use the following algorithm and define the function f on $GP(n,3)$.
step1) In the outer circle of the graph, the vertices of u_{3k+1} that $k = 0, 1, 2, ...$, are labeled by $\{1, 2, 3\}$ and the rest of the vertices are labeled of $\varnothing$. Then γ_{g3r} of the outer circle of the graph is less than or equal to $3[\frac{n}{3}] + 3$
step2) The vertices v_{3k+2} that $k = 0, 2, 4, ...$ are labeled with $\{1, 2, 3\}$. Then γ_{g3r} of these vertices is less than or equal to $\frac{3}{2}[\frac{n}{3}]$. The vertices v_{3k} that $k = 1, 3, 5, ..$ are labeled by $\{1, 2, 3\}$, then γ_{g3r} of these vertices is less than or equal to $\frac{3}{2}[\frac{n}{3}]$, and the rest of the vertices are labeled of $\varnothing$. Then γ_{g3r} of the inner round of the graph is less than or equal to $3[\frac{n}{3}]$.
Therefore $\gamma_{g3r}GP(n,3) \le 3[\frac{n}{3}] + 3 + \frac{3}{2}[\frac{n}{3}] + \frac{3}{2}[\frac{n}{3}] = 6[\frac{n}{3}] + 3$.

If $n \equiv 2, (mod3)$, we use the following algorithm and define the function f on $GP(n,3)$.
step1) In the outer circle of the graph the vertices u_{3k+1} that $k = 0, 1, 2...$, are labeled by $\{1, 2, 3\}$ and the rest of the vertices are labeled with $\varnothing$. Then γ_{g3r} the outer circle of the graph is less than or equal to $6\lceil\frac{n}{3}\rceil$
step2) The vertices v_{3k+2} that $k = 0, 2, 4, ...$ are labeled by $\{1, 2, 3\}$, then γ_{g3r} for these vertices is less than or equal to $\frac{3}{2}\lceil\frac{n}{3}\rceil$. The vertices of v_{3k} that $k = 1, 3, 5, ...$ are labeled by $\{1, 2, 3\}$ and γ_{g3r}, of these vertices is less than ir equal to $\frac{3}{2}\lceil\frac{n}{3}\rceil$.
The other vertices are labeled by $\varnothing$. Then γ_{g3r} of the inner round of the graph is less than or equal to $3\lceil\frac{n}{3}\rceil$.
Therefore $\gamma_{g3r}(GP(n,3)) \le 3\lceil\frac{n}{3}\rceil + \frac{3}{2}\lceil\frac{n}{3}\rceil + \frac{3}{2}[\frac{n}{3}] = 6\lceil\frac{n}{3}\rceil$.

□

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A Conjecture about matroid spikes

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Abstract

Spikes form an important class of 3-connected matroids. For an integer $r \geq 3$, there is a unique binary spike of rank r denoted by Z_r. By relaxing a circuit-hyperplane of Z_r, one obtains another spike and repeating this procedure produces other non-binary spikes. In this paper, we pose a conjecture which would characterize the complete set of the class of r-spikes by applying relaxing operation to all $GF(p)$-representable r-spikes, for every prime field $GF(p)$.

1 Introduction and preliminaries

Spikes play an important role in matroid theory. They have been used as a counterexample for many conjectures. So one needs to find the structure of all $\mathbb{F}$-representable spikes with rank r (It is denoted by r-spike and $r \geq 3$) in terms of matrix representation and, in particular, circuit-hyperplanes. Wu [7] evaluated the number of $GF(p)$-representable r-spikes, for p in $\{3,4,5,7\}$. He also found the asymptotic value of the number of distinct r-spikes that are representable over $GF(p)$ when p is prime. Moreover, Wu [6] showed that a $GF(p)$-representable r-spike M is only representable over fields with characteristic p provided that $r \geq 2p-1$ and M is uniquely representable over $GF(p)$. In our first paper about spikes [5], we showed that all binary spikes and many non-binary spikes of each rank can be derived from the Fano matroid by a sequence of es-splitting operations [1, 2] and circuit-hyperplane relaxations.

Let $E = \{x_1, x_2, ..., x_r, y_1, y_2, ..., y_r, t\}$ for some $r \geq 3$. Let $\mathcal{C}_1 = \{\{t, x_i, y_i\} : 1 \leq i \leq r\}$ and $\mathcal{C}_2 = \{\{x_i, y_i, x_j, y_j : 1 \leq i < j \leq r\}$. The set of circuits of every spike on E includes $\mathcal{C}_1 \cup \mathcal{C}_2$. Let $\mathcal{C}_3$ be a, possibly empty, subset of $\{\{z_1, z_2, ..., z_r\} : z_i$ is in $\{x_i, y_i\}$ for all $i\}$ such that no two members of $\mathcal{C}_3$ have more than $r-2$ common elements. Finally, let $\mathcal{C}_4$ be the collection of all $(r+1)$-element subsets of E that contain no member of $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. There is a rank-$r$ matroid M on E whose collection $\mathcal{C}$ of circuits is $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ [4] and the matroid M is called an *r-spike with tip t* and *legs* $L_1, L_2, ..., L_r$ where $L_i = \{t, x_i, y_i\}$ for all i. In an arbitrary spike M, each circuit in $\mathcal{C}_3$ is also a hyperplane of M and when such a circuit-hyperplane is relaxed, we obtain another spike. If $\mathcal{C}_3$ is empty, the corresponding spike is called the *free r-spike with tip t*.

For $r \geq 3$, let $[I_r | J_r - I_r | \mathbf{1}]$ be an $r \times (2r+1)$ matrix over $GF(2)$ whose columns are labeled, in order, $x_1, x_2, ..., x_r, y_1, y_2, ..., y_r, t$ where J_r and $\mathbf{1}$ are the $r \times r$ and $r \times 1$ matrices of all ones, respectively. This

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matrix represents the unique binary r-spike. Moreover, if we interchange $GF(2)$ and $GF(p)$, for p prime, then we obtain a matrix representation for one of $GF(p)$-representable r-spikes. But we cannot apply the es-splitting operation on such r-spikes to construct other $(r+1)$-spikes since this operation is only defined for binary matroids.

Let E be the ground set of a $GF(p)$-representable spike M. We wish to determine what subsets of E are members of $\mathcal{C}_3$ or $\mathcal{C}_4$ and how we can obtain a matrix representing M. This leads to computing the number of circuit-hyperplanes of M and producing many spikes from M which are not $GF(p)$-representable, by relaxing operation.

Let $\mathbb{F}$ be a field and $\alpha_1, \alpha_2, ..., \alpha_r$ be non-zero elements of $\mathbb{F}$. Let $\mathbf{1}$ be the $r \times 1$ matrix of all ones, and let

$$A_r = \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & \cdots & y_{r-1} & y_r \end{array} \\ \begin{pmatrix} 1+{\alpha_1}^{-1} & 1 & 1 & \dots & 1 & 1 \\ 1 & 1+{\alpha_2}^{-1} & 1 & \dots & 1 & 1 \\ 1 & 1 & 1+{\alpha_3}^{-1} & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1+{\alpha_{r-1}}^{-1} & 1 \\ 1 & 1 & 1 & \dots & 1 & 1+{\alpha_r}^{-1} \end{pmatrix} \end{array}. \tag{1}$$

Geelen, Gerards, and Whittle (2002) [3] described the following key result for the representability of spikes. Note that all circuit-hyperplanes of a r-spike are r-element subsets of form

$$\{\{z_1, z_2, ..., z_r\} : z_i \text{ is in } \{x_i, y_i\} \text{ for all } i\}.$$

So by following Proposition one can find all circuit-hyperplanes of a given r-spike.

Proposition 1.1. *Suppose that $r \geq 3$ and $\mathbb{F}$ is a field. Let M be an $\mathbb{F}$-representable r-spike with legs $\{t, x_1, y_1\}, \{t, x_2, y_2\}, ..., \{t, x_r, y_r\}$ such that $\{x_1, x_2, ..., x_r\}$ is independent. Then every $\mathbb{F}$-representation of M is projectively equivalent to a matrix of the form $[I_r|A_r|1]$ whose columns are labeled, in order, $x_1, x_2, ..., x_r, y_1, y_2, ..., y_r, t$, where A_r is as in (1). Moreover, for $K \subseteq \{1, 2, ..., r\}$, the set $\{x_k : k \notin K\} \cup \{y_k : k \in K\}$ is a circuit of M if and only if $\sum_{k \in K} \alpha_k = -1$.*

2 A conjecture about spikes

Now we propose a conjecture and a problem that may enable us to prove the conjecture. We know that when a circuithyperplane of a spike is relaxed, we obtain another spike, and repeating this procedure until all of the circuithyperplanes of a spike have been relaxed will produce the free spike. For $r \geq 3$ and some prime p, let $\mathcal{S}$ be the class of all $GF(p)$-representable r-spikes such that no two members of it are not obtained from each other by a sequence of the relaxing operations. In the following conjecture, we claim that all members of the class of all r-spikes can be constructed from all members of $\mathcal{S}$.

Conjecture 2.1. *Let $\mathcal{S}_r$ be the class of all r-spikes. Then, by applying a sequence of relaxing operations on circuit-hyperplanes of each member of $\mathcal{S}$ until we arrive at the free r-spike, we can find all members of $\mathcal{S}_r - \mathcal{S}$.*

As an example to illustrate this guess, let $r = 3$. Then $\mathcal{S} = \{F_7, P_7\}$ and in the figure below, you can see the diagram of the geometric representations of the only six 3-spikes that are obtained by sequences of relaxing operations with starting from the binary 3-spike F_7 and ternary 3-spike P_7.

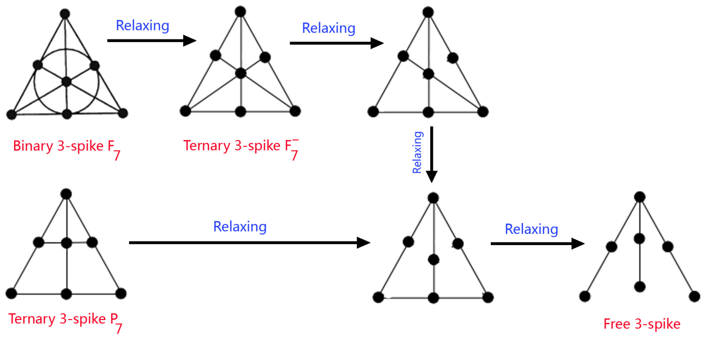

Figure 1: Geometric Representations of the only Six 3-Spikes.

Now we think that this conjecture can be proved if we can solve the following problem about relaxing operation.

Problem 2.2. *For p prime, let M be a $GF(p)$-representable matroid and let H be a circuit-hyperplane of M. Let N be a matroid that is obtained from M by relaxing H. For which filed $\mathbb{F}$ the matroid N is $\mathbb{F}$-representable?*

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The gyr-commuting graph of gyrogroups

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Abstract

The concept of gyrogroups is a natural generalization of groups and vector spaces. A gyrogroup satisfies the group condition, in addition to that, for any pair (a, b) in this structure, there exists an automorphism $gyr[a, b]$ with this property that fulfills left associativity and left loop property. The study in this article is motivated by generalizing the notion of commuting graph of groups by gyrogroups. For a gyrogroup G, the gyr-commuting graph of G, denoted by $\mathcal{GC}(G)$, is the graph with vertex set G and two distinct vertices x and y are joined by an edge if and only if $xy = gyr[x, y](yx)$. In this paper, we study some property of gyr-commuting graphs.

1 Introduction

The concept of gyrogroups, which are best motivated by the algebra of Möbius transformations of the complex open unit disc, is a natural generalization of groups and vector spaces. The resulting notions of gyrogroups and gyrovector spaces preserve the flavor of their classical counterparts and lay a fruitful bridge between non-associative algebra and hyperbolic geometry. The evolution from Möbius to gyrogroups began after the discovery that Einstein velocity addition law encodes a rich structure that is a gyrocommutative gyrogroup and a gyrovector space, in [6]. This concept is still being explored today, more than 150 years after Möbius first studied the transformations that now bear his name. Let G be a group. The commuting graph, denoted by $\mathcal{C}(G)$, is a simple graph with vertex set G and two vertices are adjacent if and only if they commute with each other. This graph was introduced by Brauer and Fowler in [3]. More studies are done in [1], [2], [4] and [5]. In this study, we introduce a structure that is a generalization of commuting graphs. We call these structures gyr-commuting graphs and several properties of this structure have been examined. Some classes of gyrosemigroups and non-gyrosemigroups of any order are introduced. Moreover, we characterize all gyrosemigroups of order two up to gyroisomorphism. In the last section, the gyrosemigroups with identity and zero are studied.

2 Main results

In this section, the commuting graph of a gyrogroup is introduced and investigated. It is shown that this notion generalized the concept of commuting graph of a group. Moreover, some properties of the graph are stated. First we recall the definition of a gyrogroup.

*Speaker

subjclass[2010]: 05C31, 05C60
keywords: Gyrogroups, Gyr-commuting graph, Commuting graph

Definition 2.1. *(Gyrogroup) (A groupoid G with a binary operation $\oplus$ is called a gyrogroup if its binary operation satisfies the following axioms:*

(1) For every $a, b \in G$, there is an $gyr[a, b] \in Aut(G, \oplus)$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b](c),$$

for every $c \in G$.

(2) For every $a, b \in G$, $gyr[a \oplus b, b] = gyr[a, b]$.

(3) There exists an element $0 \in G$, such that for all $x \in G$, $0 \oplus x = x$.

(4) For each $a \in G$, there exists $b \in G$ such that $b \oplus a = 0$.

Now, we state a lemma is needed to show the commuting graph of a gyrogroup is well-defined as a simple graph.

Lemma 2.2. *Let G be a gyrogroup. Then $x \oplus y = gyr[x, y](y \oplus x)$ if and only if $y \oplus x = gyr[y, x](x \oplus y)$.*

Following the main definition of this section is presented.

Definition 2.3. *Let G be a gyrogroup, the gyr-commuting graph of G, denoted by $\mathcal{GC}(G)$, is a simple graph whose vertices are all elements of G and two distinct vertices x and y are adjacent if and only if $x \oplus y = gyr[x, y](y \oplus x)$.*

Theorem 2.4. *For a group G, the commutating graph of G is isomorphic to the gyr-commuting graph of (G, gyr) if and only if $gyr[a, b] = identity$ for every $a, b \in G$.*

The last theorem shows that gyr-commuting graph is a generalization of commutating graph of groups. Now, we consider the vertices of $\mathcal{GC}(G)$ which are adjacent to all other vertices.

Definition 2.5. *Let G be a gyrogroup. Define gyr-center of G as follows,*

$$GC(G) = \{a \in G \mid a \oplus b = gyr[a, b]b \oplus a \ for\ all\ b \in G\ \}.$$

Lemma 2.6. *Let G be a gyrogroup. Then $GC(G) \geq 1$.*

Corollary 2.7. *For a group G, the gyr-commuting graph of G is a connected graph with diameter at most 2.*

Theorem 2.8. *For a gyrogroup G, every $a \in G$ is adjacent to a^{-1} in $\mathcal{GC}(G)$.*

Corollary 2.9. *Let G be a gyrogroup with at least 3 elements. Then the girth of $\mathcal{GC}(G)$ is equal to 3.*

Theorem 2.10. *Let G be a gyrogroup. Then the gyr-commuting graph of G is a complete graph if and only if $GC(G) = G$.*

Corollary 2.11. *For a abelian group G, if $gyr[a, b] = identity$ then $\mathcal{GC}(G)$ is a complete graph.*

In the following, the relation between the edges of $\mathcal{GC}(G)$ and $\mathcal{C}(G)$ are studied. There are some examples of groups and gyrators such that a and b are adjacent in $\mathcal{C}(G)$ but not in $\mathcal{GC}(G)$ Or vice versa. Clearly, ab is an edge of $\mathcal{C}(G)$ and $\mathcal{GC}(G)$ if and only if $b \oplus a$ is a fixed point of $gyr[a, b]$. In the next theorem more result for $gyr[a, b]$ is obtained in this case.

Theorem 2.12. *Let G be a gyrogroup. If a and b are adjacent in the both graphs $\mathcal{C}(G)$ and $\mathcal{GC}(G)$, then $gyr[a, b]$ is an automorphism of order 2.*

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k-Coalitions in Graphs

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Abstract

We propose and investigate the concept of k-coalitions in graphs, where $k \geq 1$ is an integer. A k-coalition refers to a pair of disjoint vertex sets that jointly constitute a k-dominating set of the graph, meaning that every vertex not in the set has at least k neighbors in the set. We define a k-coalition partition of a graph as a vertex partition in which each set is either a k-dominating set with exactly k members or forms a k-coalition with another set in the partition. The maximum number of sets in a k-coalition partition is called the k-coalition number of the graph represented by $C_k(G)$. We present fundamental findings regarding the properties of k-coalitions and their connections with other graph parameters, providing insights into the structural and optimization aspects of graphs with respect to the k-coalition framework.

1 Introduction

Consider a graph G with vertex set $V = V(G)$, where we only consider graphs that are simple and undirected. A coalition in G consists of two disjoint sets V_1 and V_2 of vertices, such that neither V_1 nor V_2 is a dominating set, but the union $V_1 \cup V_2$ is a dominating set of G. A *coalition partition* of G, c-partition of G for short, is a partition $\{V_1, \ldots, V_k\}$ of $V(G)$ such that for every $i \in [k]$, either the set V_i consists of a single dominating vertex of G, or V_i forms a coalition with some other part V_j. Coalition partitions were introduced in 2020 in [6] and already extensively researched in [1, 2].

Two vertices are said to be neighbors if they are adjacent. For an integer $k \geq 1$, a k-dominating set of G is a set S of vertices such that each vertex in $V \backslash S$ is adjacent to at least k vertices in S. The smallest possible size of a k-dominating set of G is referred to as the k-domination number of G, denoted by $\gamma_k(G)$. The interested reader may refer to [4, 5] for a comprehensive overview of dominating sets in graphs.

The concept of a coalition in graphs was introduced by Haynes et al. in [3]. A coalition in a graph G is a pair of sets S_1 and S_2 that are not dominating sets of G, but their union $S_1 \cup S_2$ is a dominating set of G. Such a pair forms a coalition. A vertex partition $\mathfrak{X} = \{S_1, \ldots, S_k\}$ of the vertex set $V(G)$ is called a coalition partition of G if every set $S_i \in \mathfrak{X}$ is either a dominating set of G with cardinality $|S_i| = 1$, or not a dominating set but forms a coalition with some $S_j \in \mathfrak{X}$. Haynes et al. also introduced the coalition

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keywords: Coalition partition, coalition number, dominating set

number of a graph, which is the maximum number of sets in a coalition partition. This concept has been widely studied in graph theory and has significant applications in various areas, such as wireless sensor networks, social network analysis, and distributed systems.

Motivated by the concept of coalitions in graphs, we propose and investigate the notion of k-coalitions in graphs in this paper. Two sets $U_1 \subseteq V$ and $U_2 \subseteq V$ form a k-coalition if neither is a k-dominating set, but their union is a k-dominating set. We define a k-coalition partition $\Theta = \{U_1, \ldots, U_r\}$ of a graph as a vertex partition in which each set of Θ is either a k-dominating set with exactly k members or forms a k-coalition with another set in the partition. We call the k-coalition number of a graph the maximum number of sets in a k-coalition partition denoted by $C_k(G)$.

2 Main results

In this section, we state some results.

Theorem 2.1. *For any integer $k \geq 1$ and any graph G with $\delta(G) \geq k$ there is a k-coalition partition.*

The following theorems gives the 2-coalition number of path and cycle.

Theorem 2.2. *The 2-coalition number of the path P_n is*

$$C_2(P_n) = \begin{cases} 1 & n = 1, 2, \\ 2 & n = 3, \\ 3 & n \geq 4. \end{cases}$$

Theorem 2.3. *The 2-coalition number of the cycle C_n is*

$$C_2(C_n) = \begin{cases} 4 & n \text{ is even}, \\ 3 & n \text{ is odd}. \end{cases}$$

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On diameters and domination numbers in graphs

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Abstract

In the paper [R.C. Brigham, et. al. Bicritical domination, Discrete Math, 305 (2005) 18-32] the problem; is it true that every connected bicritical graph has a minimum dominating set containing any two specified vertices of the graphs? We give a class of graphs that disprove this problem and furthermore we obtain the domination numbers and the diameters of the graphs of this class. This class of graphs has the property: $\gamma(G) - diam(G) \to \infty$ when $|V(G)| = n \to \infty$. Also for the bicritical graphs of this class $i(G) = \gamma(G)$.

1 Introduction

Let $G = (V, E)$ be a graph. A set $S \subset V$ is a dominating set if every vertex in V is either in S or is adjacent to a vertex in S, that is $V = \bigcup_{s \in S} N[s]$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G and a dominating set of minimum cardinality is called a $\gamma(G) - set$. A dominating set S is called an independent dominating set of G if there are no two vertices of S are adjacent. The minimum cardinality among the independent dominating sets of G is the independent domination number $i(G)$. We denote the distance between two vertices x and y in G by $d_G(x, y)$. Note that removing a vertex can increase the domination number by more than one, but can decrease it by at most one. The connectivity of G, written $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A graph G is k-connected if it's connectivity is at least k. A graph is k-edge-connected if every disconnecting set has at least k edges. The edge-connectivity of G, written $\lambda(G)$, is the minimum size of a disconnecting set, for more, see [10]. The circulant graph $C_{n+1}\langle 1, 4\rangle$ is the graph with vertex set $\{v_0, v_1, \dots, v_n\}$ and edge set $\{v_i v_{i+j}\ (mod\ n+1)|\ i \in \{0, 1, \dots, n\}\}$ and $j \in \{1, 4\}$. For example see figure 1, $C_{12}\langle 1, 4\rangle$.

[1] For a bicritical graph G and $x, y \in V(G)$,

$$\gamma(G) - 2 \leq \gamma(G - \{x, y\}) \leq \gamma(G) - 1.$$

*Speaker

keywords: Domination, number bicritical, diameter.

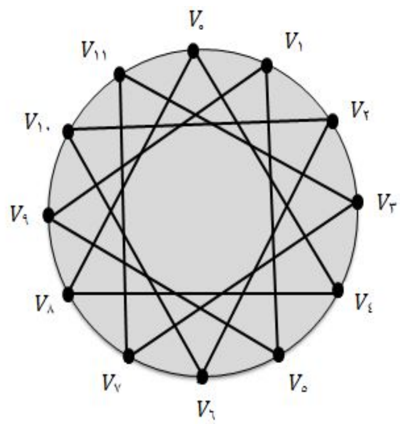

Figure 1:

2 Results

We study the domination number, diameter of $C_{n+1}\langle 1,4\rangle$ and verify their relation.

Lemma 2.1. $\gamma(C_{n+1}\langle 1,4\rangle) \le \begin{cases} 2\lfloor \frac{n}{9} \rfloor + 3 & n \equiv 7 \ (mod\ 9) \\ 2\lfloor \frac{n}{9} \rfloor + 1 & n \equiv 1 \ or\ 0 \ (mod\ 9) \\ 3 & n = 12 \\ 2\lfloor \frac{n}{9} \rfloor + 2 & o.w. \end{cases}$

Theorem 2.2. $\gamma(C_{n+1}\langle 1,4\rangle) = \begin{cases} 2\lfloor \frac{n}{9} \rfloor + 3 & n \equiv 7 \ (mod\ 9) \\ 2\lfloor \frac{n}{9} \rfloor + 1 & n \equiv 1 \ or\ 0 \ (mod\ 9) \\ 3 & n = 12 \\ 2\lfloor \frac{n}{9} \rfloor + 2 & o.w. \end{cases}$

Theorem 2.3. *The graph $C_{n+1}\langle 1,4\rangle$ is bicritical for $n+1 = 9k+3$, $n+1 = 9k+4$ or $n+1 = 9k+8$ where $k \ge 1$.*

Theorem 2.4. $\kappa(C_{n+1}\langle 1,4\rangle) = 4$*, where $n \ge 8$.*

Theorem 2.5. $\kappa'(C_{n+1}\langle 1,4\rangle) = 4$*, where $n \ge 8$.*

Theorem 2.6. *In $(C_{n+1}\langle 1,4\rangle)$ for $n \ge 7$, diam $< \gamma$.*

Corollary 2.7. *In open problem 6 of [1] we have: Is it true if G is a connected bicritical graph, then $\gamma(G) = i(G)$, where $i(G)$ is the independent domination number? We found graphs are bicritical such as $(C_{n+1}\langle 1,4\rangle)$ for $n+1 = 9k+3, 9k+4, 9k+8$ that these graphs have the property; $\gamma(G) = i(G)$.*

Now we reject the problem of [1]

Problem 2.8.

Is it true that every connected bicritical graph has a minimum dominating set containing any two specified vertices of the graphs?

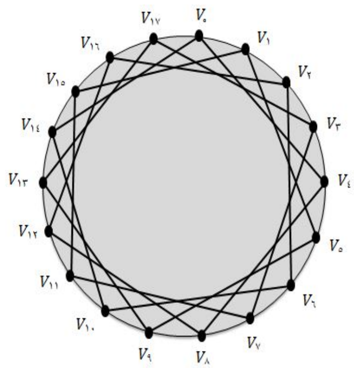

Figure 2: $(C_{18}\langle 1,4\rangle)$

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Breaking Symmetry in Graphs by Resolving Sets

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Abstract

For a given graph G, let $\dim(G)$ and $D(G)$ represent its metric dimension and distinguishing number, respectively. Here, we investigate a connection between $\dim(G)$ and $D(G)$. We show that in connected graphs, every resolving set breaks graph symmetry. Precisely, if G is a connected graph with a resolving set $S = \{v_1, v_2, \ldots, v_n\}$, then $\{\{v_1\}, \{v_2\}, \ldots, \{v_n\}, V(G) \setminus S\}$ is a partition of $V(G)$ into a distinguishing coloring, and as a consequence $D(G) \leq \dim(G) + 1$. Using this connection, certain graphs with large distinguishing number are identified.

1 Introduction

In 1977, Babai proposed a concept that today inspires many methods for distinguishing elements of graphs by automorphism. After Albertson and Collins [1] studied this concept in detail, it was widely considered in the name of *asymmetric coloring* (or *distinguishing labelling*). A distinguishing coloring of a graph is a vertex coloring such that there is no color preserving non-trivial automorphism of the graph. The minimum color required for a distinguishing coloring of a graph G is indicated by $D(G)$ and is called *distinguishing number* of G.

For a connected graph G, a subset $S \subseteq V(G)$ is a *resolving set* if for any two vertices g_1 and g_2 of G, there exists vertex $s \in S$ such that $\mathrm{d}(g_1, s) \neq \mathrm{d}(g_2, s)$. The smallest size that can be taken by a resolving set S is called the *metric dimension* and is denoted by $\dim(G)$. The concept of resolving sets was introduced independently and simultaneously by Slater [4] and Harary & Melter [3] in 1975-6.

These two concepts have gone their research paths for years without paying attention to each other. Here we show that there is a connection between them. In fact, we will show that solving the metric dimension problem breaks the symmetry in graphs. We construct graphs G such that $D(G) = n$ and $\dim(G) = m$ for all values of n and m, where $1 \leq n < m$. Furthermore, we have characterized all graphs G of order n with $D(G) \in \{n-1, n-2\}$. For any graph G, let $G_c = G$ if G is connected, and $G_c = \overline{G}$ if G is disconnected. Let G^* denote the twin graph obtained from G by contracting any maximal set of vertices with the same open or close neighborhood into a vertex. Let $\mathcal{F}$ be the set of all graphs except graphs G with the property that $\dim(G_c) = |V(G)| - 4$, $\mathrm{diam}(G_c) \in \{2, 3\}$ and $5 \leq |V(G_c^*)| \leq 9$. We characterize all graphs $G \in \mathcal{F}$ of order n with the property that $D(G) = n - 3$.

*Speaker

subjclass[2010]: 05C15.
keywords: Breaking symmetry, resolving sets, distinguishing number, metric dimension

Let $(v_1, v_2, \dots, v_n)$ denoted the path P_n on the vertices $v_1, v_2, \dots, v_n$. Let C_5' denote the C_5 with a chord. All definitions and symbols used and undefined are standard and can be found in [2].

2 Main results

In this section, we show that in connected graphs, the distinguishing number is bounded by the metric dimension plus one. In addition, we characterize some graphs with large distinguishing number.

Theorem 2.1. *Let G be a connected graph. Then $D(G) \leq \dim(G) + 1$.*

Note that this bound is sharp. For instance, let $G = P_n$ for an integer $n \geq 2$. Hence, $D(G) = \dim(G) + 1 = 2$.

Corollary 2.2. *Let G be a connected graph. For any resolving set $S = \{v_1, v_2, \dots, v_n\}$ of G, and $\{\{v_1\}, \{v_2\}, \dots, \{v_n\}, V(G) \setminus S\}$ is a partition of $V(G)$ into a distinguishing coloring.*

In the next theorem, we construct graphs G such that $D(G) = n$ and $\dim(G) = m$ for all values of n and m, where $1 \leq n < m$.

Theorem 2.3. *For every $1 \leq n < m$, there exists a graph G having distinguishing number n and metric dimension m.*

In the next two theorems, we have characterized all graphs G of order n with $D(G) \in \{n-1, n-2\}$.

Theorem 2.4. *Let G be a graph with order n. Then $D(G) = n-1$ if and only if G is one of the following:*

(1) C_4

(2) $K_{t,1}$, $t \geq 2$

(3) $2K_2$

(4) $K_t \cup K_1$, $t \geq 2$

Theorem 2.5. *Let G be a graph with order $n \geq 4$. Then $D(G) = n-2$ if and only if G is one of the following:*

(1) C_5

(2) P_4

(3) $K_{1,2,2}$

(4) $2K_2 \cup K_1$

(5) $K_{3,3}$

(6) $2K_3$

(7) $K_{t,2}$, $t \geq 3$

(8) $K_t \cup K_2$, $t \geq 3$

(9) $K_2 + \overline{K_t}, t \geq 2$

(10) $K_t \cup 2K_1$, $t \geq 2$

(11) $K_t + \overline{K_2}$, $t \geq 2$

(12) $\overline{K_t} \cup K_2$, $t \geq 2$

(13) $K_1 + (K_t \cup K_1)$, $t \geq 2$

(14) $K_{t,1} \cup K_1$, $t \geq 2$

Let G be a graph of order n with vertices $v_1, v_2, \dots, v_n$, and assume that $H_1, H_2, \dots, H_n$ are complete or empty graphs. The *blow-up* of G, denoted by $G[H_1, H_2, \dots, H_n]$, is the graph obtained as follows: (1) every vertex v_i of G is replaced by H_i for every i with $1 \leq i \leq n$, (2) for any two vertices v_i and v_j if $v_i v_j \in E(G)$, then for every $u \in V(H_i)$ and every $v \in V(H_j)$, uv is an edge of $G[H_1, H_2, \dots, H_n]$. Specifically, in the path $(v_1, v_2, \dots, v_n)$, $P_n[H_1, \dots, H_n] = (H_1, H_2, \dots, H_n)$.

Theorem 2.6. *Let G be a graph of order $n \geq 5$. Let $\mathcal{F}$ be the set of all graphs except graphs G with the property that* $\dim(G_c) = n-4$, $\operatorname{diam}(G_c) \in \{2,3\}$ *and* $5 \leq |V(G_c^*)| \leq 9$. *If $G \in \mathcal{F}$, then $D(G) = n-3$ if and only if G is one of the following:*

(1) P_5

(2) $C_5^{'}$

(3) $K_{4,4}$

(4) $2K_4$

(5) $K_3 + \overline{K_t}$, $t \geq 3$

(6) $\overline{K_3} \cup K_t$, $t \geq 3$

(7) $K_2 + (K_t \cup K_1)$, $t \geq 2$

(8) $\overline{K_2} \cup K_{t,1}$, $t \geq 2$

(9) $K_{t,3}$, $t \geq 4$

(10) $K_t \cup K_3$, $t \geq 4$

(11) $K_t + \overline{K_3}$, $t \geq 3$

(12) $\overline{K_t} \cup K_3$, $t \geq 3$

(13) $K_t + (K_2 \cup K_1)$, $t \geq 2$

(14) $\overline{K_t} \cup K_{2,1}$, $t \geq 2$

(15) $K_{1,2,t}$, $t \geq 3$

(16) $K_1 \cup K_2 \cup K_t$, $t \geq 3$

(17) $K_2 + K_{2,2}$

(18) $\overline{K_2} \cup 2K_2$

(19) $K_{1,3,3}$

(20) $2K_3 \cup K_1$

(21) $K_{2,2,2}$

(22) $3K_2$

(23) $\overline{K_2} + (K_1 \cup K_t)$, $t \geq 2$

(24) $K_2 \cup K_{t,1}$, $t \geq 2$

(25) $\overline{K_2} + 2K_2$

(26) $K_2 \cup K_{2,2}$

(27) $\overline{K_t} + (K_1 \cup K_2)$, $t \geq 2$

(28) $K_t \cup K_{2,1}$, $t \geq 2$

(29) $K_2 + 2K_2$

(30) $2K_1 \cup K_{2,2}$

(31) $K_1 + (K_1 \cup K_{1,t})$, $t \geq 2$

(32) $K_1 \cup (K_1 + (K_t \cup K_1))$, $t \geq 2$

(33) $K_1 + P_4$

(34) $K_1 \cup P_4$

(35) $K_1 + (K_1 \cup 2K_2)$

(36) $K_1 \cup (K_1 + K_{2,2})$

(37) $K_1 + (K_1 \cup K_{2,2})$

(38) $K_1 \cup (K_1 + 2K_2)$

(39) $P_4[K_1, K_t, K_1, K_1]$, $t \geq 2$

(40) $P_4[\overline{K_t}, K_1, K_1, K_1]$, $t \geq 2$

(41) $P_4[K_1, \overline{K_t}, K_1, K_1]$, $t \geq 2$

(42) $P_4[K_t, K_1, K_1, K_1]$, $t \geq 2$

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The relation between perfect star packing and efficient dominating set in fullerene graph

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Abstract

Fullerene graph is a connected plane cubic graph with only pentagonal and hexagonal faces, which is the molecular graph of carbon fullerene. A perfect star packing in a graph G is a spanning subgraph of G whose every component is isomorphic to the star graph $K_{1,3}$. Many fullerene graphs arise from smaller fullerene graphs by applying some transformations. An efficient dominating set of graph G is a vertex subset D of G such that each vertex of G not in D is adjacent to exactly one vertex from D and any two vertices of D are not adjacent in G. Clearly, a perfect star packing in a fullerene graph G on n vertices will exist if and only if G has an efficient dominating set of cardinality $\frac{n}{4}$.

1 Introduction

A fullerene is a spherically shaped molecule consisting of carbon atoms in which every carbon ring forms a pentagon or a hexagon A fullerene graph is a planar, cubic, 3-connected graph with only pentagonal and hexagonal faces. An important result derived straightforward from the Euler relation, $|V| + |F| = |E| + 2$, where $|V|$ is the number of vertices, $|F|$ is the number of faces and $|E|$ is the number of edges of G, ensures that every fullerene graph has exactly 12 pentagonal faces, A perfect star packing of a fullerene graph G is of type P_0 if all the centers of stars lie on hexagons of G Otherwise it is of type P_1. Every atom of a fullerene has bonds with exactly three neighboring atoms. dominating set of a graph G is a set of vertices D such that each verte in $V(G) - D$ is adjacent to a vertex in D. Moreover, if each vertex in $V(G) - D$ is adjacent to exactly one vertex in D and D is an independent vertex set, then D is called efficient.

2 Main results

The problem of determining the existence of efficient dominating sets in some families of graphs was first investigated by Biggs [2] and Kratochvil [3] Later Livingston and Stout [4] studied the existence and

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construction of efficient dominating sets in families of graphs arising from th interconnection networks of parallel computers. The problem of finding an efficient dominating set, however, is algorithmically hard [5].

Theorem 2.1 ([8])**.** *Let S be a perfect star packing of fullerene graph G. Then $G - C(S)$ has even number of odd cycles.*

Proof. Proof. If $G - C(S)$ does not have a non-facial cycle of G, then any pentagon of G does not have a vertex in $C(S)$. So all the vertices on pentagons are leaves in S. It implies that $G - C(S)$ has exactly twelve odd cycles, each of which is a pentagon. Next, we suppose that $G - C(S)$ has a non-facial cycle of G, denoted by C.
Claim 1. If C is an even cycle, then G has even number of pentagons which share edges with C. If C is an odd cycle, then G has odd number of pentagons which share edges with C. the number of pentagons which share edges with C is equal to $n_2. n_2$ and the length of C have the same parity. So the Claim holds.
Claim 2. Any pentagon of G shares edges with at most one non-facial cycle in $G - C(S)$. Let P be a pentagon of G. P has at most one vertex which is the center of a star in S. If P does not have a vertex in $C(S)$, then P is a cycle in $G - C(S)$. each component of $G - C(S)$ is an induced cycle of G. So P does not share edges with any non-facial cycle in $G - C(S)$. If P has a vertex $x \in C(S)$, then $P - x$ is a subgraph of a non-facial cycle in $G - C(S)$. So P shares edges with exactly one non-facial cycle in $G - C(S)$.
Now, we consider the following two cases for the non-facial cycles in $GC(S)$.
Case 1. $G - C(S)$ does not have a non-facial cycle of odd length. Then any non-facial cycle C in $G - C(S)$ is of even length. By the above Claims, there are even number of pentagons in G such that they share edges with C. Since G has exactly twelve pentagons, there are even number of pentagons in G each of which does not share edges with non-facial cycles in $G - C(S)$. These pentagons must be cycles in $G - C(S)$ Hence $G - C(S)$ has even number of odd cycles.
Case 2. $G - C(S)$ has some non-facial cycle of odd length. Suppose that $G - C(S)$ has exactly k non-facial cycles of odd length. We denote the number of pentagons in G each of which does not share edges with non-facial cycles in $G - C(S)$ by p. These p pentagons must be cycles in $GC(S)$. So $G - C(S)$ has $p + k$ odd length cycles. Next, we show that p and k have the same parity. If p is odd, then G has odd number of pentagons each of which share edges with exactly one non-facial cycle in $G - C(S)$ since G has exactly 12 pentagons. By the above Claims, for each even length non-facial cycle in $G - C(S)$, G has even number of pentagons which share edges with the cycle, and for each odd length non-facial cycle in $G - C(S)$, G has odd number of pentagons which share edges with the cycle. So $G - C(S)$ has odd number of non-facial cycles of odd length. This means that k is odd. For p being even, we can similarly show that k is even. So k and p have the same parity and $p + k$ is even. □

Theorem 2.2. *If fullerene graph G has a perfect star packing, then the order of G is divisible by 8.*

Proof. Proof. We suppose that S is a perfect star packing of G and $\mathcal{C}_o$ and $\mathcal{C}_e$ are the collections of all the odd cycles and even cycles in $G - C(S)$, respectively. Then we have the following equation.

$$\begin{aligned} |V(G)| &= |C(S)| + \sum_{C \in \mathcal{C}_o} |C| + \sum_{C \in \mathcal{C}_e} |C| \\ &= \frac{|V(G)|}{4} + \sum_{C \in \mathcal{C}_o} |C| + even \end{aligned} \tag{1}$$

By Theorem 2.1 $\mathcal{C}_o$ has even number of elements. Combine the above equation, we know that $\frac{|V(G)|}{4} \times 3$ is even. Hence $\frac{|V(G)|}{4}$ is even, that is, the order of G is divisible by 8. □

Theorem 2.3. *The order of a fullerene graph with an efficient dominating set is $8n$.*

Proof. From the definitions of the efficient dominating set and the perfect star packing of a fullerene graph G, it is a natural result, a fullerene graph G with n vertices has a perfect star packing if and only if G has an efficient dominating set of cardinality $\frac{n}{4}$ so with the theorem 2.2, we get that The order of a fullerene graph with an efficient dominating set is 8n. □

Clearly, for a fullerene graph G with a perfect star packing, its order must be divisible by 4. So the order of G is 8k or $8k + 4$ for some positive integer k. So the order of G can not be $8k + 4$. Theorem 2.2 is equivalent to the following corollary.

Corollary 2.4. *A fullerene graph with order* $8n + 4$ *does not have a perfect star packing.*

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Groups with the same enhanced power graphs

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Abstract

The enhanced power graph $P_e(G)$ of a group G is a graph with vertex set G, where two vertices u and v are adjacent if and only if $< u, v >$ is cyclic. In this paper, we raise and study the following question: For which natural numbers n every two groups of order n with isomorphic enhanced power graphs are isomorphic?

1 Introduction

The concept of a power graph was introduced in the context of semigroup theory by Kelarev and Quinn [1]. Let G be a finite group, the undirected power graph $P(G)$ is the undirected graph with vertex set G, where two vertices $a, b \in G$ are adjacent if and only if $a \neq b$ and $a^m = b$ or $b^m = a$ for some positive integer m. Likewise, the directed power graph $\overrightarrow{P}(G)$ is the directed graph with vertex set G, where for two vertices $u, v \in G$ there is an arc from a to b if and only if $a \neq b$ and $b = a^m$ for some positive integer m. In this paper we study the same power graphs and enhanced power graphs. Clearly $G_1 \cong G_2$ implies $P(G_1) \cong P(G_2)$. Does the converse hold? The converse has been considered and it's is false for finite groups in general. For example, if p is an odd prime and $m > 2$, besides the elementary abelian group H of order p^m, there are non-abelian groups G of order p^m and exponent p, so H and G are non-isomorphic but have isomorphic power graphs. On the other hand, if both G and H are abelian then $P(G) \cong P(H)$ implies $G \cong H$.

In [3], the following question was investigated and the set of all such numbers was denoted by S:

For which natural numbers n every two groups of order n with the same spectrum are isomorphic?

In [2], we raise another question along the same lines:

For which natural numbers n, every two groups of order n with isomorphic power graphs are isomorphic?

Let us denote the set of all such numbers by $\bar{S}$. Since two finite groups with isomorphic power graphs have the same spectrum, it is easy to see that $S \subseteq \bar{S}$.

In this paper we pay attention to the same question for enhanced power graphs.

Question: For which natural numbers n, every two groups of order n with isomorphic enhanced power graphs are isomorphic?

Let us denote the set of all such numbers by $\bar{\bar{S}}$. it is easy to see that $\bar{\bar{S}} = \bar{S}$.

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2 Main results

We know if enhanced power graphs are isomorphic then power graphs are isomorphic, so groups have the same number of elements of each order. Therefore, if $P_e(G) \cong P_e(H)$ then G and H have the same number of elements of each order. By expanding these explanations, we reach the following Theorem.

Theorem 2.1. *If G is one of the following finite groups:*

1. *A cyclic group,*
2. *A symmetric group,*
3. *A dihedral group,*
4. *A generalized quaternion group,*

and H is a finite group such that $P_e(G) \cong P_e(H)$ implies that $G \cong H$.

In [2], we have proved some results for the set $\bar{S}$. Since $\bar{S} = \bar{\bar{S}}$, all those results hold for the set $\bar{\bar{S}}$.

Proposition 2.2. *If p is a prime number, then $2p^2 \in \bar{\bar{S}}$.*

Theorem 2.3. *If $n \notin \bar{\bar{S}}$ and $(n,k) = 1$, then $nk \notin \bar{\bar{S}}$.*

Theorem 2.4. *Let $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ $(r \geq 0)$. If $\alpha_0 \geq 4$ or there exists $i \neq 0$ such that $\alpha_i \geq 3$, then $n \notin \bar{\bar{S}}$.*

Corollary 2.5. *Every odd element of $\bar{\bar{S}}$ is cube-free.*

Theorem 2.6. *The set $\bar{\bar{S}} \setminus S$ is non-empty. Its smallest element is 72.*

For the set $\bar{S} \setminus S$, we tried to generalize the example of groups of orders 72. If we could prove for every odd prime p, the number $8p^2 \in \bar{S} \setminus S$, then the set $\bar{S} \setminus S$ is infinite. Define the following two groups:

$G = \langle a, b, x, y \mid a^p = b^p = x^2 = y^2 = 1, (xa)^2 = (xb)^2 = 1, ab = ba, ay = ya, by = yb, (xy)^4 = 1 \rangle$.

$G' = \langle a, b, x, y \mid a^p = b^p = x^2 = y^2 = z^2 = 1, (xy)^2 = z, ab = ba, xax = a^{-1}, xbx = b, yay = b, yby = a \rangle$.

We checked that G and G' are conformal and $P(G')$ cannot be isomorphic with $P(G)$. This is not enough to prove that for every odd prime p, $8p^2 \in \bar{S} \setminus S$, because there might be further pairs of conformal groups of the given order. However, we suspect that such pairs do not exist, which led us to the following conjecture:

Conjecture 2.7. *The set $\bar{S} \setminus S$ is infinite.*

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Properties of Sombor Index Matrix of Some Graphs

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Abstract

In this article, we study the Sombor index matrix of some graphs and find determinant of them by the determinant of adjacency matrices.

1 Introduction

One of topological indices introduced by Ivan Gutman in 2021 name the Sombor index. This index use for predicting some phisicochemical properties of substances.

Definition 1.1. *Let $G = (V, E)$ be a graph such that $V(G)$ denotes vertices set and $E(G)$ denotes edges set, then the adjacency matrix of G, $A(G) = (a_{ij})_{nn}$ is defined as bellow*

$$a_{ij} = \begin{cases} 1 & if\ v_i\ and\ v_j\ are\ adjacent \\ 0 & otherwise \end{cases}$$

and the Sombor matrix of G, $\mathcal{A}_{SO} = a'_{ij}$ is defined as below

$$a'_{ij} = \begin{cases} \sqrt{d_i^2 + d_j^2} & if\ v_i\ and\ v_j\ are\ adjacent \\ 0 & otherwise \end{cases} \tag{1}$$

Also Sombor index of graph G is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d^2u + d^2v}. \tag{2}$$

Theorem 1.2. *let G is a complete graph with n vertices, then*

$$det(\mathcal{A}_{SO}(K_n)) = [\sqrt{2}(n-1)]^n(-1)^{n-1}(n-1) = \sqrt{2}^n(-1)^{n-1}(n-1)^{n+1}. \tag{3}$$

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subjclass[2010]: 05C31, 05C60
keywords: Sombor index matrix, Determinant of Sombor index matrix, Toeplitz matrix

Proof. By this propertiy that $det(aA(G)) = a^n det(A(G))$ and $det(A(K_n)) = (-1)^{n-1}(n-1)$, it is concluded that

$$det(\mathcal{A}_{SO}(K_n)) = [\sqrt{2}(n-1)]^n(-1)^{n-1}(n-1) = \sqrt{2}^n(-1)^{n-1}(n-1)^{n+1}. \tag{4}$$

Remark 1.3. *Let G, be a r -regular graph and A(G) and $\mathcal{A}_{SO}(G)$ denote the adjacency matrix and Sombor matrix of graph G, in respectively, then, $\mathcal{A}_{SO}(G) = \sqrt{2}rA(G)$ and*

$$det(\mathcal{A}_{SO}(G)) = (\sqrt{2}r)^n det(A(G)). \tag{5}$$

□

Definition 1.4. *A Toeplitz matrix is an $n \times n$ matrix $T_n = [t_{k,j}, k, j = 0, \dots n-1]$ where $t_{k,j} = t_k - j$, in other words, a matrix of the form*

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \cdots & \\ t_2 & t_1 & t_0 & \ddots & \\ \vdots & \vdots & \vdots & & \vdots \\ & & & & \end{bmatrix} \tag{6}$$

Theorem 1.5. *Let G is a complete graph with n vertices, then*

$$\mathcal{A}_{SO}(G).A(G) = T_n \tag{7}$$

where T_n is a Toeplitz matrix such that

$$t_{k,j} = \begin{cases} \sqrt{2}(n-1)^2 & k = j \\ \sqrt{2}(n-1)(n-2) & k \neq j \end{cases} \tag{8}$$

and

$$\mathcal{A}_{SO}(G)^2 = T_n \tag{9}$$

where T_n is a Toeplitz matrix such that

$$t_{k,j} = \begin{cases} 2(n-1)^3 & k = j \\ 2(n-1)^2(n-2) & k \neq j \end{cases} \tag{10}$$

Proof. By multiple the Sombor matrix in own and comparing it with the Toeplitz matrix. □

Theorem 1.6. *Let G be a cycle graph with n vertices, then*

$$det(\mathcal{A}_{SO}(C_n)) = \begin{cases} 2^{n+1}(\sqrt{2})^n, & \textit{if n is odd} \\ 0, & \\ -2^{n+2}(\sqrt{2})^n, & \end{cases}$$

Proof. The determinant of $A(C_n)$ for $n \geq 3$ is $\det(\mathrm{A}(\mathrm{C}_n)) = \begin{cases} 2, & \text{if n is odd} \\ 0, & \\ -4, & \end{cases}$ So, $\det(\mathcal{A}_{SO}(C_n)) =$

$$(\sqrt{8})^n det A(C_n) = \begin{cases} 2^{n+1}(\sqrt{2})^n, & \text{if n is odd} \\ 0, & \\ -2^{n+2}(\sqrt{2})^n, & \end{cases}$$

□

Theorem 1.7. *If $K(m,n)$, $m,n \in N$ be a complete bipartite graph, then*

$$\mathcal{A}_{SO}(K(m,n)) = aA(K(m,n)) \tag{11}$$

where $a = \sqrt{m^2+n^2}$. Also

$$det(\mathcal{A}_{SO}(K(m,n))) = a^n det(A(K(m,n))) = 0. \tag{12}$$

Proof. Whereas the columns of the matrix A is linear dependent, so the determinant of the matrix A is zero and in result $det(\mathcal{A}_{SO}(K(m,n))) = 0$. □

Corollary 1.8. *If S_n, $n \in N$ be a star graph, then*

$$\mathcal{A}_{SO}(S_n) = aA(S_n) \tag{13}$$

where $a = \sqrt{(n-1)^2+1} = \sqrt{n^2-2n+2}$. Also

$$det(\mathcal{A}_{SO}(S_n)) = a^n det(S_n)) = 0. \tag{14}$$

Proof. Whereas the columns of the matrix A is linear dependent, so the determinant of the matrix A is zero and in result $det(\mathcal{A}_{SO}(S_n)) = 0$. □

2 Conclusions

In this paper, we provid the Sombor index matrix for some graphs and find the determinant of them by the determinant of the adjacency matrices of the graphs.

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Sombor index of some cactus chain graphs

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Abstract

In this paper, we study the Somber index of some cactus chain graphs without any pendant edges and some pendants edges.

1 Introduction

In the references [1] , [3] , [4] and [5] the Sombor index of trees and some cactus chain graphs has calculated. and in [2] Zagreb index formula stated as below and we use it for gaining the Sombor index.

Let $G = (V, E)$ be a graph such that $|V(G)| = n$ and $|E(G)| = m$, then the Somber index and Zagreb index are respectively defined as following:

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d^2(u) + d^2(v)}, \tag{1}$$

$$M_1(G) = \sum_{uv \in E(G)} (d(u) + d(v)) = \sum_{u \in V(G)} d^2(v) \tag{2}$$

where $d(u)$ is degree of vertex u in G, A non-trival connected graph in which each edge locates in at most one cycle is named a cactus gragh.

2 The Sombor Index of Some Cactus Chain Graphs

Theorem 2.1. *Let G be a cactus chain of polygonals with the sides number $p_i > 3$, then $SO(G) = 2\sqrt{2}\sum_{i=1}^{c} p_i - 8(\sqrt{2} - \sqrt{5})(c-1)$ ans $SO(G) = \frac{M_1(G)}{\sqrt{2}} + (c-1)\frac{8\sqrt{10}-24}{\sqrt{2}}$.*

Proof. mathematical induction on c. □

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keywords: Sombor Index, Cactus Chain, Zagreb Index

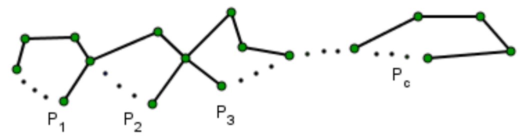

Figure 1: A cactus chain graph G with ideal p_i- polygonals $(p_i > 3)$ and c ring.

Corollary 2.2. *Let $G = (V(G), E(G))$ be a cactus chain such that $|V(G)| = n, |E(G)| = m$, p be the number of the sides of a polygonal $(p \geq 3)$ and c be the number of the rings in the cactus chain as the figures 2, then*

$$SO(G) = \begin{cases} 6\sqrt{2} & p = 3,\ c = 1 \\ 4(\sqrt{2}+\sqrt{5})c - 4\sqrt{2} & p = 3,\ c > 1 \\ 2\sqrt{2}p + 2(c-1)[(p-4)\sqrt{2} + 4\sqrt{5}] & \forall\ p > 3 \text{ and } \forall\ c. \end{cases} \tag{3}$$

Also

$$SO(G) = \begin{cases} \frac{M1(G)}{\sqrt{2}} = 6\sqrt{2} & p = 3,\ c = 1 \\ \frac{M_1(G)}{\sqrt{2}} + \frac{8\sqrt{10}-24}{\sqrt{2}} + (\frac{4\sqrt{10}-12}{\sqrt{2}})(c-2) & p = 3,\ c \geq 2 \\ \frac{M1(G)}{\sqrt{2}} + \frac{8\sqrt{10}-24}{\sqrt{2}}(c-1) & p > 3,\ \forall c. \end{cases} \tag{4}$$

Proof. By the mathematical induction on c and p. □

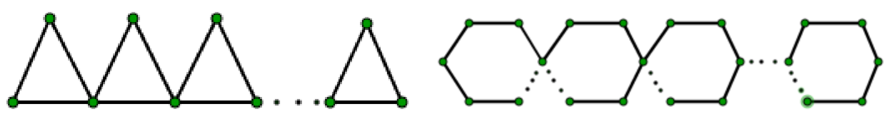

Figure 2: A cactus chain graph G for $p = 3$ and $p \geq 4$ and c ring.

Theorem 2.3. *let G_0 be a connected graph contained vertices u_1, u_2, u_3, u_4 such that $d_{G_0}(u_1) = 2, d_{G_0}(u_2) = 3 \vee 4, d_{G_0}(u_3) = 2, d_{G_0}(u_4) = 2 \vee 4$ and $\{u_1u_2, u_1u_3, u_3u_4\} \subseteq E(G_0)$. Suppose that $L = v_1v_2...v_l$ is a path. Denote by G_1 the graph gotten from G_0, L by attaching vertices u_1v_1. Let $G_2 = G_1 - u_1v_1 + u_3v_1$. Then*

$$\begin{cases} SO(G_2) \geq SO(G_1), & if\ d_{G_0}(u_4) \geq d_{G_0}(u_2) \\ SO(G_2) < SO(G_1), & if\ d_{G_0}(u_4) < d_{G_0}(u_2). \end{cases} \tag{5}$$

Proof. By the definition of the Sombor index, $SO(G_2) - SO(G_1) = (\sqrt{2^2+3^2} + \sqrt{3^2 + d^2_{G_0}(u_4)}) - (\sqrt{3^2 + d^2_{G_0}(u_2)} + \sqrt{3^2+2^2})$, then consider probable numbers for $d_{G_0}(u_4)$ and $d_{G_0}(u_2)$ and determine the sign of the relation for them. □

Theorem 2.4. *Let $G = (V(G), E(G))$, $|V(G)| = n$, $|E(G)| = m$ and p be the number of sides of a poliygonal of the cactus chain such that for $p > 3$ and $c > 3$ every two rings put on non-adjacent vertices* *and the other vertices have pendant one edge, then*

$$SO(G) = \begin{cases} 9\sqrt{2} + 3\sqrt{10} & c = 1,\ p = 3 \\ (4\sqrt{2} + \sqrt{10} + 10)c + 2(\sqrt{10} - \sqrt{2}) & c > 1,\ p = 3 \\ \big(3\sqrt{2} + p\sqrt{10}\big)c + (20 - 2\sqrt{10} - 12\sqrt{2})(c - 1) & \forall c,\ p > 3. \end{cases} \tag{6}$$

Also

$$SO(G) = \begin{cases} \frac{M1(G)}{\sqrt{2}} + \frac{3\sqrt{20}-12}{2} & p = 3, c = 1 \\ \frac{M1(G)}{\sqrt{2}} + \frac{4\sqrt{20}+20\sqrt{2}-44}{\sqrt{2}} + (c-2)\frac{10\sqrt{2}+\sqrt{20}-18}{\sqrt{2}} & p = 3, c \geq 2 \\ \frac{M1(G)}{\sqrt{2}} + (\sqrt{18} + \sqrt{10} - 5\sqrt{2})p + (c-1)\frac{20\sqrt{2}+2\sqrt{20}-36}{\sqrt{2}} & \\ +(p-4)(c-1)\frac{3\sqrt{2}+\sqrt{10}-5\sqrt{2}}{\sqrt{2}} & p \geq 4, \forall c \end{cases} \tag{7}$$

Proof. By the mathematical induction on c and p. □

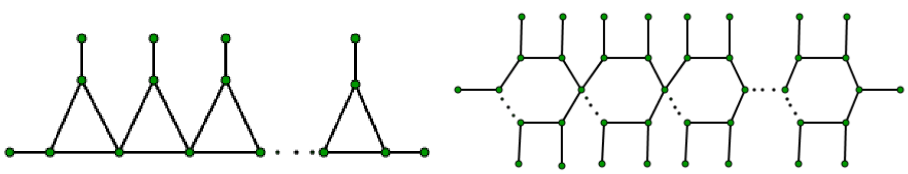

Figure 3: Graph G for $p = 3$ and $p \geq 4$

3 Conclusions

In this paper, we provid the Sombor index formula directly and undirectly (base on the Zagreb index) for a group of the chain graphs.

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Independent domination gap of a graph

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Abstract

A dominating set in a graph $G = (V, E)$ is a set S such that every vertex of G is either in S or adjacent to a vertex in S. The minimum cardinality of a dominating set in G is called the domination number of G denoted by $\gamma(G)$, and the maximum cardinality of a minimal dominating set in G is called the upper domination number of G and is denoted by $\Gamma(G)$. The difference between these two parameters is called the domination gap of G and is denoted by $\mu_d(G) = \Gamma(G) - \gamma(G)$. A graph G with $\mu_d(G) = 0$ is called a well-dominated graph. Motivated by well-dominated graph, we introduce well-independent dominated graphs. A graph G with $\mu_{id}(G) = \Gamma_i(G) - \gamma_i(G) = 0$ is called a well-independent dominated graph, where $\gamma_i(G)$ and $\Gamma_i(G)$ are independent domination number and upper independent domination number of G, respectively. We obtain some results about the independent domination gap in special graphs.

1 Introduction

All graphs considered in this paper are finite undirected graphs without loop nor parallel edge. Let G be such a graph. A non-empty set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G) \backslash S$ is adjacent to at least one vertex in S. The minimum cardinality of all dominating sets of G is called the *domination number* of G and is denoted by $\gamma(G)$. For a detailed treatment of domination theory, the reader is referred to [4]. The maximum cardinality of a minimal dominating set in G is called the upper domination number of G denoted by $\Gamma(G)$. The difference between these two parameters is called the domination gap of G and denote it by $\mu_d(G) = \Gamma(G) - \gamma(G)$. A graph G with $\mu_d(G) = 0$ is called a well-dominated graph. Domination gap was first introduced by Finbow et al. in [2]. An independent dominating set of G is a vertex subset that is both dominating and independent in G, or equivalently, is a maximal independent set. The independent domination number of G, denoted by $\gamma_i(G)$, is the minimum size of all independent dominating sets of G. The following relationship between these parameters in consideration is well-known [4],

$$\gamma(G) \leq \gamma_i(G) \leq \alpha(G).$$

The maximum cardinality of a minimal independent dominating set in G is called the upper independent domination number of G denoted by $\Gamma_i(G)$. For more information on independent dominating sets and their counting refer to [3, 5].

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*Speaker

subjclass[2010]: 05C69

keywords: Domination, independent domination, well independent dominated

Motivated by the domination gap, we introduce well-independent dominated graphs and obtain some results about this kind of graphs.

2 Main results

In this section, we introduce well-independent dominated graphs and obtain some calculation of the independent domination gap in special graphs.

Definition 2.1. *The difference between the upper independent domination number of G and the independent domination number of G is called the independent domination gap of G and denote it by $\mu_{id}(G) = \Gamma_i(G) - \gamma_i(G)$. We say G is a well-independent dominated graph, if $\mu_{id}(G) = 0$.*

The friendship (or Dutch-Windmill) graph F_n is a graph that can be constructed by the coalescence of n copies of the cycle graph C_3 of length 3 with a common vertex. Let n and $q \geq 3$ be any positive integer and $F_{q,n}$ be the *generalized friendship graph* formed by a collection of n cycles (all of order q), meeting at a common vertex. The n-book graph ($n \geq 2$) is defined as the Cartesian product $K_{1,n} \square P_2$. We call every C_4 in the book graph B_n, a page of B_n. All pages in B_n have a common side v_1v_2. The following lemma gives the independent domination number of the friendship graph and the book graph.

Lemma 2.2. *(i) If F_n is a friendship graph, then $\mu_{id}(F_n) = n - 1$.*

(ii) For $k = 4, 5, 6$, $\mu_{id}(F_{k,n}) = n\lfloor \frac{k}{3} \rfloor - 1$.

(iii) If B_n is a book graph, then $\mu_{id}(B_n) = 1$.

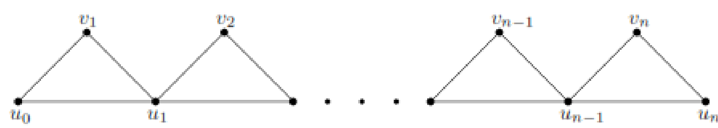

Figure 1: Chain triangular cactus T_n

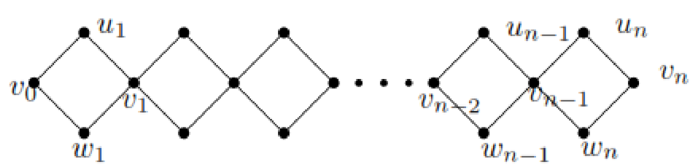

Figure 2: Para-chain square cactus Q_n

We consider a class of simple linear polymers called cactus chains. Cactus graphs were first known as Husimi tree, they appeared in the scientific literature some sixty years ago in papers by Husimi and Riddell concerned with cluster integrals in the theory of condensation in statistical mechanics. We refer the reader to papers [1] for some aspects of parameters of cactus graphs.

A cactus graph is a connected graph in which no edge lies in more than one cycle. Consequently, each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus G are cycles of the same size i, the cactus is i-uniform. A triangular cactus is a graph whose blocks are triangles, i.e., a 3-uniform cactus. A vertex shared by two or more triangles is called a cut-vertex. If each triangle of a triangular cactus G has at most two cut-vertices, and each cut-vertex is shared by exactly two triangles, we say that G is a chain triangular cactus (Figure 1). we denote the chain triangular cactus of length n by T_n. By replacing triangles in this definitions with cycles of length 4 we obtain cacti whose every block is C_4. We

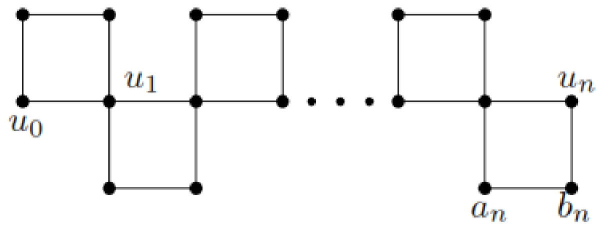

Figure 3: Ortho-chain square cactus O_n

call such cacti square cacti. Note that the internal squares may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-square (Figure 3); if the cut-vertices are not adjacent, we call the square a para-square (Figure 2).

Here we compute the independent domination gap of some cactus chains:

Theorem 2.3. *(i)* $\mu_{id}(T_n) = \lfloor \frac{n}{2} \rfloor$.

(ii) $\mu_{id}(Q_n) = n - 1$.

(iii) *For* $n \geq 2$, $\mu_{id}(O_n) = 1$.

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Global Dominator Coloring of Graphs

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Abstract

Let $G = (V, E)$ be a graph and $D \subseteq V$. A vertex $v \in V$ is a dominator of D if v dominates every vertex in D and v is said to be an anti-dominator of D, if v dominates none of the vertices of D. Let $C = (V_1, V_2, ..., V_k)$ be a coloring of G. A color class V_i is called a dom-color class or an anti dom-color class of the vertex v according as v is a dominator of V_i or an anti-dominator of V_i. The coloring C is called a global dominator coloring of G, if every vertex of G has a dom-color class and an anti dom-color class in C. The minimum number of colors required for a global dominator coloring of G is called the global dominator chromatic number and is denoted by $\chi_{gd}(G)$. In this paper, we study the global dominator chromatic number for some graphs.

1 Introduction

Let G be a simple graph. For any vertex $v \in V(G)$, the open neighborhood of v is the set $N(v) = \{u \in V(G) | u \sim v\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $D \subset V$, the open neighborhood of D is $N(D) = \bigcup_{v \in D} N(v)$ and the closed neighborhood of S is $N[D] = N(D) \cup D$. A set $D \subset V$ is a dominating set if $N[D] = V$, or equivalently, every vertex in $V \setminus D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of the dominating set in G. A dominating set D of G is a global dominating set of G if D is also a dominating set of the complement G^c of G. The minimum cardinality of a global dominating set of G is called the global domination number of G and is denoted by $\gamma_g(G)$ or simply γ_g.

Some concepts (such as dominator coloring and total dominator coloring) related to domination and coloring have been introduced and well-studied in the literature. A dominator coloring of a graph G is a coloring of G in which every vertex dominates every vertex of at least one color class. A total dominator coloring of G, is a proper coloring of the vertices of G in which each vertex of the graph is adjacent to every vertex of some color class. The minimum number of colors required for a dominator coloring of G (total dominator coloring) is called the dominator chromatic number of G and is denoted by $\chi_d(G)$ (by $\chi_d^t(G)$) (see [1, 5]).

A vertex $v \in V$ is a dominator of D if v dominates every vertex in D and v is said to be an anti-dominator of D, if v dominates none of the vertices of D. Let $C = (V_1, V_2, ..., V_k)$ be a coloring of G.

*Speaker

subjclass[2010]: 05C15, 05C69

keywords: global domination, coloring, global dominator coloring

A color class V_i is called a dom-color class or an anti dom-color class of the vertex v according as v is a dominator of V_i or an anti-dominator of V_i. The coloring C is called a global dominator coloring of G, if every vertex of G has a dom-color class and an anti dom-color class in C. The minimum number of colors required for a global dominator coloring of G is called the global dominator chromatic number and is denoted by $\chi_{gd}(G)$. This parameter has introduced by Hamid and Rajeswari in [3]. They have determined the value of χ_{gd} for some common classes of graphs such as paths, cycles, complete multipartite graphs and the Petersen graph. As a vertex v dominates itself, the vertex v is a dominator of $\{v\}$, whereas it is not an anti-dominator of $\{v\}$. Hence a graph G does not admit a global dominator coloring when $\Delta(G) = n-1$. For example the friendship graph F_n which is join of K_1 and nK_2 does not admit a global dominator coloring.

In this paper, we continue the study of the global dominator chromatic number and compute it for some graphs.

2 Main results

In this section, determine the value of χ_{gd} for some graphs. We need the following lemmas:

Lemma 2.1. *(i) For the path P_n on $n \geq 4$ vertices, we have*

$$\chi_{gd}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1; & \text{if } n = 7, \\ \\ \lceil \frac{n}{3} \rceil + 2; & \text{elsewhere.} \end{cases}$$

(ii) For the cycle C_n on $n \geq 4$ vertices, $\chi_{gd}(C_n) = \lceil \frac{n}{3} \rceil + 2$.

Theorem 2.2. *[4] If G is a triangle-free graph, then $\gamma \leq \gamma_g \leq \gamma + 1$*

Theorem 2.3. *[2] Let T be a tree. Then $\gamma_g(T) = \gamma + 1$ if and and only if either T is a star or T is a tree of diameter 4 which is constructed from two or more stars, each having at least two pendant vertices, by connecting the centres of these stars to a common vertex.*

Theorem 2.4. *For two paths P_n, P_m on $n, m \geq 2$ vertices, we have*

$$\chi_{gd}(P_n \square P_m) = \begin{cases} \left\lceil \dfrac{nm}{3} \right\rceil + 1 & if\ \ n = 2, m = 4, \\ \\ \left\lceil \dfrac{nm}{3} \right\rceil + 2 & elsewhere. \end{cases}$$

The following theorem gives the global dominator chromatic number of corona of P_n and K_1, i.e., $P_n \circ K_1$.

Theorem 2.5. $\chi_{gd}(P_n \circ K_1) = n + 1$.

By virtue of a theorem, the family of trees can be split into two classes, namely Class 1 and Class 2. A tree T is of Class 1 or Class 2 according as $\chi_{gd}(T) = \gamma_g(T) + 1$ or $\chi_{gd}(T) = \gamma_g(T) + 2$. The problem of characterizing trees of Class 1 or Class 2 seems to be little challenging ([3]). By Theorem 2.5 we see that the graph $P_n \circ K_1$ is an example of trees T for which $\chi_{gd}(T) = \gamma_g(T) + 1$.

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Some results on the Sombor index and energy of graphs

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Abstract

For a graph G, the Sombor index $SO(G)$ is a recently introduced by I. Gutman. In this talk, we provide some lower and upper bounds for $SO(G)$ and improve the known bounds for the Sombor index.

Joint work with: S. Akbari and M. Habibi

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ and $E(G)$ are the vertex set and the edge set of G, respectively. For a vertex $v \in V(G)$, we denote the degree of v by d_v. By an r-regular graph, we mean a graph in which all of its edges have the same degree. Also, the maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. In this talk, the *energy* of a graph G, is shown by $\mathcal{E}(G)$ and is defined as the sum of the absolute values of its adjacency eigenvalues. Let C be a subset of $V(G)$. If every edge of $E(G)$ is incident with a vertex of C, then the set C is said to be a *vertex cover set* of G. Moreover, the *Sombor index* of G is defined as $SO(G) = \sum_{xy \in E(G)} \sqrt{d_x^2 + d_y^2}$.

Definition 1.1. [1] *For a graph G with vertex set $v_1, \dots, v_n$, the energy of the vertex v_i with respect to G, which is denoted by $\mathcal{E}(v_i)$, is given by*

$$\mathcal{E}(v_i) = \textstyle\sum_{i=1}^{n} |A(G)|_{ii} \qquad \text{for } i = 1, \dots, n,$$

where $|A| = (AA^*)^{1/2}$ and A is the adjacency matrix of G.

Therefore, according to the definition of energy of graphs, we have:

$$\mathcal{E}(G) = \mathcal{E}(v_1) + \cdots + \mathcal{E}(v_n).$$

Also,

$$\mathcal{E}(G) = \textstyle\sum_{xy \in E(G)} \left(\frac{\mathcal{E}(x)}{d_x} + \frac{\mathcal{E}(y)}{d_y}\right).$$

*Speaker

subjclass[2010]: 05C31, 05C60
keywords: Sombor index, energy graph

In fact, we can define the energy of the edge $e = xy$ by decomposing it into two adjacent vertices x and y, as follows:

$$\mathcal{E}(e) = \frac{\mathcal{E}(x)}{d_x} + \frac{\mathcal{E}(y)}{d_y}.$$

Theorem 1.2. *[1, Pro. 3.2] For a graph G and a vertex $x \in V(G)$, we have $\mathcal{E}(x) \leq \sqrt{d_x}$ with equality if and only if the connected component containing v_x is isomorphic to the star graph S_n and x is its center.*

Theorem 1.3. *[1, Pro. 3.3] Let G be a connected graph with at least one edge. Then $\mathcal{E}(x) \geq \frac{d_x}{\Delta(G)}$, for all $x \in V(G)$. Equality holds if and only if G is isomorphic to complete bipartite graph $K_{d,d}$.*

Theorem 1.4. *[1, Thm. 3.6] Let G be a graph with at least one edge. Then $\mathcal{E}(v_i) \geq \sqrt{\frac{d_i}{\Delta(G)}}$, for every $v_i \in V(G)$.*

Recall that a graph G is called *completely non-hypoenergetic* if $\mathcal{E}(v) \geq 1$ for all $v \in V(G)$. Clearly, according to the above theorem, regular graphs are completely non-hypoenergetic. This result is stated in Proposition 4.9 of [1].

Theorem 1.5. [1] *Let G be a graph with vertex covering set C. Then $\sum_{x \in C} \mathcal{E}(x) \geq \frac{1}{2}\mathcal{E}(G)$.*

2 Main results

In Theorem 5 of [2], the authors proved the following result.

Theorem 2.1. *Let G be a simple graph. If C is a vertex-covering set of G, then*

$$\frac{\sqrt{\delta(G)}\mathcal{E}(G)}{2} + \frac{|C|\Delta^2(G)}{\sqrt{2}} \leq SO(G)$$

In this talk we improve this result as follows:

Theorem 2.2. *Let G be a graph. If C is a vertex-covering set of G, then*

$$\mathcal{E}(G) \leq \frac{2}{\delta(G)\sqrt{\delta(G)}} SO(G) + 4\frac{\sqrt{\delta(G)} - 1}{\delta(G)} |E(G[C])|.$$

In Theorem 7 of [3] the following upper bound for $SO(G)$ is stated in terms of $\mathcal{E}(G)$, $\Delta(G)$ and size of G as follows:

Theorem 2.3. *Let G be a graph of size m. Then $SO(G) \leq \sqrt{\mathcal{E}(G) m \Delta^5(G)}$*

Now, we improve this upper bound for regular graphs, as follows:

Theorem 2.4. *Let G be an r-regular graph of order n. Then $SO(G) \leq \sqrt{\frac{nr^5}{2}\mathcal{E}(G)}$.*

In [2] the authors proved the following result for bipartite graphs.

Theorem 2.5. *[2, Thm. 4] Let G be a bipartite graph. Then $\mathcal{E}(G) \leq \sqrt{\frac{2}{\delta^3(G)}} SO(G)$.*

Now, in the following theorem, we prove this bound for an arbitrary graph.

Theorem 2.6. *Let G be graph. Then $\mathcal{E}(G) \leq \sqrt{\frac{2}{\delta^3(G)}} SO(G)$. Moreover, the equality holds if and only if each connected component of G is isomorphic to K_2.*

We finish this work by stating two new following lower bounds for $SO(G)$.

Theorem 2.7. *Let G be a graph. Then we have* $SO(G) \geq \delta(G)\sqrt{\mathcal{E}(G)}$.

Theorem 2.8. *Let G be a graph of size m . Then* $SO(G) \geq \delta(G)\sqrt{\mathcal{E}(G) - \frac{2m\Delta(G)}{\delta(G)}}$.

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On the Ramsey number of stars versus an odd wheel

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Abstract

For given graphs $G_1, G_2, \ldots, G_t$ the Ramsey number $R(G_1, G_2, \ldots, G_t)$ is the smallest positive integer n such that if the edges of the complete graph K_n are partitioned into t disjoint color classes giving t graphs $H_1, H_2, \ldots, H_t$, then at least one H_i has a subgraph isomorphic to G_i. In this note, the exact value of $R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}, W_m)$ will be determined for odd m, $m \le \Sigma_{i=1}^{t}(n_i - 1) + 1$.

1 Introduction

In this note, we only concerned with undirected simple finite graphs and we follow [1] for terminology and notations not defined here. As usual, a complete graph, a cycle, a path and a star on n vertices are denoted by K_n, C_n, P_n and $K_{1,n-1}$, respectively. The *wheel* W_m is the graph on $m+1$ vertices obtained from the cycle C_m by adding one vertex x, called the *hub* of the wheel, and making x adjacent to all vertices of C_m, called the *rim* of the wheel. The wheel W_m is called odd if m is odd.

Let $G, G_1, G_2, \ldots, G_t$ be given simple graphs. The Ramsey number $R(G_1, G_2, \ldots, G_t)$ is the smallest positive integer n such that if the edges of the complete graph K_n are partitioned into t disjoint color classes giving t graphs $H_1, H_2, \ldots, H_t$, then at least one H_i has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by the Ramsey's classical result [6]. There is very little known about $R(G_1, G_2, \ldots, G_c)$ for $c \ge 3$, even for very special graphs. For a survey on Ramsey theory, we refer the reader to the regularly updated survey by Radziszowski [5]. The Ramsey number of stars was determined in [2], as follows.

Theorem 1.1. ([2]) *If $R = R(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t})$ and $\Sigma = \Sigma_{i=1}^{t}(n_i - 1)$, then*

(i) *$R = \Sigma + 2$ if either Σ is odd or Σ is even and all n_i's are odd.*

(ii) *$R = \Sigma + 1$ if Σ is even and at least one n_i is even.*

A graph G is called *H-free* if it does not contains H as a subgraph. The notation $ex(p, H)$ is defined as the maximum number of edges in a H-free graph on p vertices. The exact value of the $ex(p, C_n)$ is known in some cases. The following theorem can be found in [1].

*Speaker

subjclass[2010]: 05C55
keywords: Ramsey number, Turán number, Tree, Wheel

Theorem 1.2. ([1]) *If n and p are positive integers such that $n \le \frac{1}{2}(p+3)$, then $ex(p, C_n) = \lfloor \frac{p^2}{4} \rfloor$.*

In 1959, Erdős and Gallai [3] proved that every graph G on p vertices and at least $\frac{(n-2)}{2}p + 1$ edges contains a path of order n, i.e $ex(p, P_n) \le \frac{(n-2)}{2}p$. Motivated by this result, Erdős and Sós conjectured that if G is a graph on p vertices and more than $\frac{(n-2)}{2}p$ edges, then G contains every tree T on n vertices. In other words, $ex(p, T) \le \frac{(n-2)}{2}p$. Various specific cases of this conjecture have already been proved, especially for trees of diameter at most four [4].

2 Main results

The aim of this note is to determine the Ramsey number $R(K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_t}, W_m)$, for odd m, $m \le \Sigma_{i=1}^{t}(n_i - 1) + 2$. Hereafter, for given positive integers $n_1, n_2, \dots, n_t$ we use Σ to denote $\Sigma_{i=1}^{t}(n_i - 1)$. In the following theorem, we determine $R(K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_t}, C_m)$ for odd m, $m \le \Sigma + 1$.

Theorem 2.1. *Let m be odd, $T_{n_1}, \dots, T_{n_t}$ are trees satisfying Erdős and Sós conjecture. If $r = R(T_{n_1}, T_{n_2}, \dots, T_{n_t})$ and $m \le \Sigma + 1 \le r$, then $R(T_{n_1}, T_{n_2}, \dots, T_{n_t}, C_m) = 2r - 1$.*

Proof. By the definition, there exists a t-edge coloring, say c, of K_{r-1} such that the i-th color class, $1 \le i \le t$, contains no copy of T_i. Let A and B be two disjoint copies of K_{r-1} whose edges are colored by t colors $\alpha_1, \alpha_2, \dots, \alpha_t$ according to c. Now, color the remaining edges of $2K_{r-1}$ (edges between A and B) by an additional color α_{t+1}. Clearly, the induced graph on edges with color α_{t+1} is a bipartite graph and so can not contain C_m, because m is odd. This observation shows that $R \ge 2r - 1$.

Now, assume that K_{2r-1} is edge-colored by colors $\alpha_1, \alpha_2, \dots, \alpha_{t+1}$ and let H_i, $1 \le i \le t+1$, denote the subgraph of K_{2r-1} induced by edges with color α_i. Since $T_{n_1}, T_{n_2}, \dots, T_{n_t}$ are trees satisfying Erdős and Sós conjecture, using Theorem 1.2, we may assume that

$$|E(H_i)| \le \frac{n_i - 1}{2}(2r-1), \quad |E(H_{t+1})| \le \left\lfloor \frac{(2r-1)^2}{4} \right\rfloor.$$

Using Theorem 1.1, one can easily check that $\sum_{i=1}^{t+1} |E(H_i)| < |E(K_{2r-1})|$ for $m \le \Sigma \le r+1$, which means that $R \le 2r - 1$. This observation completes that proof. □

Finally, using Theorem 2.1, we have the following theorem, which determine the exact value of the $R(K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_t}, W_m)$ for odd m, $m \le \Sigma + 1$.

Theorem 2.2. *If m is odd, $\Sigma \ge m - 2$ and $r = R(K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_t})$, then*

$$R(K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_t}, W_m) = 3r - 2.$$

Proof. By the definition, there exists a t-edge coloring of K_{r-1}, say c, such that the i-th color class, $1 \le i \le t$, contains no copy of K_{1,n_i}. Let A, B and C be three disjoint copies of K_{r-1} whose edges are colored by t colors $\alpha_1, \alpha_2, \dots, \alpha_t$ according to c. Now, color the remaining edges of $3K_{r-1}$ (edges between A, B and C) by an additional color α_{t+1}. Clearly, the induced graph on edges with color α_{t+1} is tripartite and so can not contain W_m, because $\chi(W_m) = 4$. This observation shows that $R \ge 3r - 2$.

Now, consider an arbitrary $(t+1)$-edge coloring of $G = K_{3r-2}$ by colors $\alpha_1, \alpha_2, \dots, \alpha_{t+1}$ and let H_i, $1 \le i \le t+1$, be the subgraph of K_{3r-2} induced by the edges of color α_i. We assume that $K_{1,n_i} \nsubseteq H_i$, $1 \le i \le t$, and we prove that $W_m \subseteq H_{t+1}$. Let H be the subgraph of K_{3r-2} induced by the edges with colors $\alpha_1, \alpha_2, \dots, \alpha_t$.

Claim. $\delta(H) \le r - 2$.

On the contrary, let $\delta(H) \ge r - 1$. If either Σ is odd or Σ is even and all n_i are odd, then by Theorem 1.1, $r = \Sigma + 2$ and so $\delta(H) \ge \Sigma + 1$, which means that $K_{1,n_i} \subseteq H_i$ for some i, $1 \le i \le t$, a contradiction. Thus, let Σ be even and at least one n_i, say n_t, be even. In this case, by Theorem 1.1 $r = \Sigma + 1$ and so each vertex of H must have degree precisely Σ and each color appears exactly $n_i - 1$ times in each vertex of H, because $K_{1,n_i} \notin H_i$, $1 \le i \le t$. Now, H_t is a $(n_t - 1)$-regular graph on $3r - 2$ vertices. Since Σ and n_t are even, we are seeking a regular graph of odd order and degree, a contradiction. This contradiction shows that $\delta(H) \le r - 2$.

Let v be a vertex in H with $\deg_H(v) \le r - 2$ and $G' = G - (N(v) \cup \{v\})$. Clearly G' has at least $2r - 1$ vertices and so by Theorem 2.1, we have a copy of C_m in color α_{t+1} in G' and hence a copy of W_m in H_{t+1} with the hub v. This observation shows that $R \le 3r - 2$ which completes that proof. □

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On the generalization of Cayley graphs

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Abstract

Let G be a group and S be a symmetric subset of G ($i.e.$, $S^{-1} = S$). The Cayley graph $\mathrm{Cay}(G, S)$ is the graph whose vertices are the elements of G and two distinct vertices g and h are adjacent if and only if $g^{-1}h \in S$. Now, let H be a subgroup of G and S be a subset of G where $S \cap H$ is a symmetric subset of H. The group-subgroup pair graph, or briefly pair graph, which is denoted by $\mathrm{Cay}(G, H, S)$, is the undirected graph with vertices G and simple edges given by (h, hs) for all $h \in H, s \in S$. If we set $H = G$, then the pair graph $\mathrm{Cay}(G, H, S)$ is isomorphic to the Cayley graph $\mathrm{Cay}(G, S)$ which means that the pair graph is a generalization of Cayley graphs. In this paper, we study the local metric dimension of $\mathrm{Cay}(G, H, S)$. Also, we investigate the metric dimension, local metric dimension and edge metric dimension of $\mathrm{Cay}(\mathbb{Z}_n, H, S)$.

1 Introduction

Let G be a group and S be a symmetric subset of G ($i.e.$, $S^{-1} = S$). The Cayley graph $\mathrm{Cay}(G, S)$ is the graph whose vertices are the elements of G and two distinct vertices g and h are adjacent if and only if $g^{-1}h \in S$. Cayley graphs have been widely investigated, for example in [1] and [5]. Recently, in [4], for a group-subgroup pair (G, H) and a subset S of G where $S \cap H$ is a symmetric subset of H, a Cayley-type graph which is a generalization of Cayley graphs, was introduced and studied. This graph is called the group-subgroup pair graph, or briefly pair graph, and we denote it by $\mathrm{Cay}(G, H, S)$ in this paper. $\mathrm{Cay}(G, H, S)$ is the undirected graph with vertices G and simple edges given by (h, hs) for all $h \in H, s \in S$.

Let G be a connected and simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $u, v \in V(G)$. The distance between u and v, which is denoted by $d(u, v)$, is the length of a shortest path connecting them. Let $W = \{v_1, \ldots, v_k\}$ be a subset of $V(G)$. Then the vector $r(v|W) = (d(v, v_1), \ldots, d(v, v_k))$ is the metric W-representation of a vertex $v \in V(G)$. A subset $W \subseteq V(G)$ is a metric generator for G if the vertices of G have pairwise different metric W-representations. A metric basis for G, is a metric generator of the smallest order, and its order is the metric dimension $\dim(G)$ of G. The definition of metric dimension was proposed by F. Harary and R. A. Melter [3]. Usually, it is not needed to distinguish

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all pairs of vertices but only adjacent ones. From this reason the local metric dimension was introduced in [2]. A subset $W \subseteq V(G)$ is called a local metric generator for G if the adjacent vertices of G have pairwise different metric W-representations. The local metric dimension of G, which is denoted by $\dim_\ell(G)$, is the smallest order of a local metric generator, and such a set is a local metric basis for G. The vector $r(e|W) = (d(e, v_1), \ldots, d(e, v_k))$ is the edge metric W-representation of an edge $e \in V(G)$, where for an edge $e = xy \in E(G)$ and $v \in V(G)$, we have $d(e, v) = \min\{d(x, v), d(y, v)\}$. A subset $W \subseteq V(G)$ is an edge metric generator for G if the edges of G have pairwise different edge metric W-representations. An edge metric generator of the smallest order is called an edge metric basis for G, and its order is called the edge metric dimension of G, which is denoted by $\mathrm{edim}(G)$ [6].

2 Main results

In this Section, we study the local metric dimension of $\mathrm{Cay}(G, H, S)$. Also, we investigate the metric dimension, local metric dimension and edge metric dimension of $\mathrm{Cay}(\mathbb{Z}_n, H, S)$, when $H = \langle p\rangle$ for a prime number p, and $S = \{\pm1, \pm3\}$.

Proposition 2.1. *Assume that Cay(G, H, S) is connected. Then*

$$\dim_\ell(\mathit{Cay}(G, H, S)) \leqslant |H|.$$

Also if S contains exactly one representative from each coset of H other than H, where $H \neq \{e\}$, then $\dim_\ell(\mathit{Cay}(G, H, S)) \leqslant |H| - 1$.

Proposition 2.2. *Let H be a nontrivial subgroup of G such that Cay(G, H, S) be connected and $S_H = \varnothing$. Then*

$$\dim_\ell(\mathit{Cay}(G, H, S)) \leqslant |H| - 1.$$

Moreover if S contains exactly one representative from each coset of H other than H, then $\dim_\ell(\mathit{Cay}(G, H, S)) = 1$.

Example 2.3. *Let G be a group of order n and H be a subgroup of G with $S_H = \varnothing$.*
(i) Let $H = \{e\}$. Then Cay(G, H, S) is the union of $K_{1,|S|}$ and $|G| - |S| - 1$ isolated vertices. So $\dim_\ell(\mathit{Cay}(G, H, S)) = 1$.
(ii) Let $H = G$. Then S is the empty set and so Cay(G, H, S) is the empty graph. Thus $\dim_\ell(\mathit{Cay}(G, H, S)) = 1$.
(iii) Let $S = G - H$. Then since $S_H = \varnothing$, there is no adjacency between the vertices of H. Also each vertex in H is adjacent to all vertices of S, and each vertex in $G - H$ is adjacent to all vertices of H. So Cay(G, H, S) is isomorphic to the complete bipartite graph $K_{|S|,|H|}$, and therefore we have $\dim_\ell(\mathit{Cay}(G, H, S)) = 1$.

In the following theorem, we investigate the local metric dimension of $\mathrm{Cay}(G, H, S)$ in a general situation. To do this, for a subset S of $G - H$, we set $S^{\sim} := H \cap SS^{-1}$.

Theorem 2.4. *Assume that G is a group, H is a subgroup and S is a subset of G. Let $K = < S_H \cup S_O^{\sim} >$. Then*

$$\dim_\ell(\mathit{Cay}(G, H, S)) = [H : K] \dim_\ell(\Gamma_e),$$

where Γ_e is the induced subgraph of Cay(G, H, S) with vertex set $K \cup KS_O$.

Example 2.5. *Suppose that $G = \mathbb{Z}_{55}$, $H = \langle 5\rangle$ and $S = \{\pm1, \pm3\}$. Since $S_H = \varnothing$ and $S_O = S$, we have $S_O^{\sim} = H \cap SS^{-1} = \{0\}$ and $K = \langle \varnothing \cup \{0\}\rangle = \langle 0\rangle$. Hence $K \cup KS_O = S \cup \{0\}$. Since Γ_e is the induced subgraph of Cay$(\mathbb{Z}_{55}, \langle 5\rangle, S)$ with vertex set $S \cup \{0\}$, we have Γ_e is isomorphic to $K_{1,4}$. So $\dim_\ell(\Gamma_e) = 1$. Clearly $[H : K] = 11$. Therefore, by Theorem 3.4 we have $\dim_\ell(\mathit{Cay}(\mathbb{Z}_{55}, \langle 5\rangle, S)) = [H : K] \dim_\ell(\Gamma_e) = 11 \times 1 = 11$.*

Proposition 2.6. *Let $G = \mathbb{Z}_n$, H be a proper subgroup of G and $S = \{\pm1\}$. Then Cay(G, H, S) is either a union of $|H|$ copies of P_3 with $n - 3|H|$ isolated vertices, or it is the cycle C_n.*

Corollary 2.7. *Let $G = \mathbb{Z}_n$, $S = \{\pm1\}$ and H be a proper subgraph of G. If $2 \notin H$, then we have*

1. $\dim(Cay(\mathbb{Z}_n, H, S)) = |H|$, *if* $n = 3|H|$. *Otherwise,* $\dim(Cay(\mathbb{Z}_n, H, S)) = n - 2|H| - 1$.
2. $\dim_\ell(Cay(\mathbb{Z}_n, H, S)) = |H|$.
3. $edim(Cay(\mathbb{Z}_n, H, S)) = |H|$.

Also if $2 \in H$, *then the following statements hold.*

1. $\dim(Cay(\mathbb{Z}_n, H, S)) = 2$.
2. $\dim_\ell(Cay(\mathbb{Z}_n, H, S)) = 1$ *if* n *is even. Otherwise* $\dim_\ell(Cay(\mathbb{Z}_n, H, S)) = 2$.
3. $edim(Cay(\mathbb{Z}_n, H, S)) = 2$.

Theorem 2.8. *Let* $n \equiv 1(mod 6)$ *and* p *be a prime number that divides* n. *Assume that* $H = \langle p \rangle$ *and* $S = \{\pm 1, \pm 3\}$. *Then the following statements hold.*

1. $\dim(Cay(\mathbb{Z}_n, H, S)) = n - |H| - 1$.
2. $\dim_\ell(Cay(\mathbb{Z}_n, H, S)) = |H|$.
3. $edim(Cay(\mathbb{Z}_n, H, S)) = 4|H|$.

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Perfect 1-factorisations of complete k-uniform hypergraphs

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Abstract

A 1-factorisation of a graph is called perfect if it satisfies each of the following equivalent conditions: 1) the union of each pair of 1-factors is isomorphic to the same connected subgraph, 2) the union of each pair of 1-factors is connected and 3) the union of each pair of 1-factors is a Hamilton cycle. A 1-factorisation of a graph is called uniform if the union of each pair of 1-factors is isomorphic to the same subgraph. In this paper we generalise the concept of uniform 1-factorisations from graphs to hypergraphs in the natural way, and based on the three conditions above, we define four generalisations of perfect 1-factorisations of graphs to the context of hypergraphs and obtain some results.

1 Introduction

A 1-regular spanning subgraph of a graph is known as a 1-factor. A partition of the edge set of a graph G into $alpha$ 1-factors is called a 1-factorisation of G often denoted by $F = \{F_1, ..., F_\alpha\}$. A natural question is: under what conditions does a 1-factorisation for the complete graph on n vertices , K_n? Clearly n must be even, and one of the earliest proofs that this condition is sufficient is Kirkmans 1847 construction of 1-factorisations of K_n for all even integers $n \geq 2$ [1]. Given a 1-factorisation of a graph G, a well-studied problem is to ask if the 1- factorisation has the property that the union of each pair of 1-factors is isomorphic to the same subgraph H of $G\dot{S}$uch a 1-factorisation is called a uniform 1-factorisation (U1F) of G and the subgraph H is called the common graph. Furthermore, a uniform 1-factorisation in which the common graph is a Hamilton cycle is called a perfect 1- factorisation (P1F)$\dot{T}$he following famous conjecture is due to Kotzig [2].

Conjecture 1.1. For any $n \geq 2$, K_{2n} admits a perfect 1-factorisation.
Kotzig provided an infinite family of 1-factorisations of the complete graph K_{2n} that are perfect when $2n - 1$ is an odd prime. [3]
Bryant , Maenhaut, and Wanless constructed another infinite family of P1Fs of K_{2n} where $2n - 1$ is an odd prime, which is not isomorphic to the family given by Kotzig.[4]
Anderson gave an infinite family of 1-factorisations of K_{2n} that are perfect when n is an odd prime. [5]

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Besides these infinite families there are a number of sporadic values of n such that K_{2n} has been shown to admit a P1F. Most recently a P1F of K_{56} was found by Pike which leaves K_{64} as the smallest complete graph for which the existence of a P1F is unknown; [6] for more information on the orders of complete graphs with known P1Fs, a paper on the number of non-isomorphic P1Fs of K_{16} by Gill and Wanless is recommended. [7]

For uniform 1-factorisations that are not perfect, the common graph will be a collection of two or more disjoint cycles of even lengths . We say that a U1F has type $(c_1, c_2, ..., c_t)$ if the common graph of the U1F is a collection of t cycles of lengths $c_1, c_2, ..., c_t$. For complete graphs K_{2n} with $2n \leq 16$, all types of U1Fs up to isomorphism are known due to a result by Meszka and Rosa [8].

Theorem 1.2. If F is a U1F of K_{2n}, where $2n \leq 16$, then F is one of the following:
(a) a P1F;
(b) a U1F of K_8 of type (4, 4);
(c) a U1F of K_{10} of type (4, 6);
(d) a U1F of K_{12} of type (6, 6);
(e) a U1F of K_{16} of type (4, 4, 4, 4).
Further, the U1Fs from cases (b), (c), (d), (e) are unique up to isomorphism.

Besides the above U1Fs there are several known infinite families of U1Fs; for further information on these families the survey paper on P1Fs by Rosa is recommended. [9]

The goal of this paper is to generalise the concepts of uniform and perfect 1- factorisations from graphs to hypergraphs. A hypergraph H consists of a non-empty vertex set $V(H)$ and an edge set $E(H)$ where each element of $E(H)$ is a non-empty subset of the vertex set $V(H)$. The complete k-uniform hypergraph of order n, denoted K_n^k is the hypergraph on n vertices, where the edges are precisely all the k-sets of the vertex set. In this paper , to avoid the case of graphs we will consider only k-uniform hypergraphs for $k \geq 3$.

2 Main results

1) The generalization of the concept of 1-factor and 1-factorization from graphs to hypergraphs is relatively simple.
2) The 2-factor of a hypergraph is a 1-regular sub-supergraph , and the decomposition of a hypergraph into 1-factors is a separate edge. It is a factorization
3) The necessary condition for the existence of 1-uniform k-complete hypergaf factorization on n vertices is that $k|n$.
   and Baranyain showed that this condition is also sufficient for $k \geq 3$.

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Some bounds for the energy of a graph and Sombor index

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Abstract

The energy $\mathcal{E}(G)$ of a graph G is the sum of the absolute values of all the eigenvalues of its adjacency matrix $A(G)$. The Sombor index of G is defined as $\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}$, where d_u and d_v are the degrees of vertices u and v in G. For any graph G we state an upper bound for $SO(G)$ in terms of size of G. Moreover, we find some new bounds for the Sombor index in terms of graph energy and consider the relationship between them.

Joint work with: S. Akbari and M. Habibi

1 Introduction

Let $G=(V(G),E(G))$ be a simple graph, where $V(G)$ and $E(G)$ denote the set of its vertices and edges, respectively. By the *size* of G, we mean the number of its edges. The maximum and minimum degrees of G are denoted by Δ and δ, respectively. A path of order n is denoted by P_n. The *adjacency matrix* of G, denoted by $A(G)$, is an $n\times n$ matrix whose (i,j)-entry is 1 if v_i and v_j are adjacent and 0 otherwise. The *energy* of a graph G, is shown by $\mathcal{E}(G)$ and is defined as the sum of the absolute values of its adjacency eigenvalues, i.e.

$$\mathcal{E}(G)=\sum_{i=1}^{n}|\lambda_i|$$

where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of $A(G)$. This concept was introduced by I. Gutman and is intensively studied in chemistry, since it can be used to approximate the total π-electron energy of a molecule (see, e.g.[6],[7]). This graph invariant has been intensively studied in the last two decades, and plenty of research papers on this subject can be found in the literature in the field of applied mathematics and mathematical chemistry. The Sombor index $SO(G)$ of G is defined as $\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}$, where d_u and

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d_v is the degrees of vertices u and v in G. In [3] the energy of P_n was calculated as follows:

$$\mathcal{E}(P_n) = \begin{cases} \dfrac{2}{sin\dfrac{\pi}{2(n+1)}} - 2 & if\ n \equiv 0 \pmod 2 \\ \\ \dfrac{2cos\dfrac{\pi}{2(n+1)}}{sin\dfrac{\pi}{2(n+1)}} - 2 & if\ n \equiv 1 \pmod 2 \end{cases}.$$

Theorem 1.1. *[2] For a graph G with vertices $v_1, \ldots, v_n$ we have*

$$\mathcal{E}(G) \le \sum_{i=1}^{n} \sqrt{d_i} \le \sqrt{2mn}.$$

Theorem 1.2. *[4] Let G be a connected graph of order n. If $n \ge 3$ then $\mathcal{E}(G) < SO(G)$.*

Theorem 1.3. *[11] Let G be a graph of size m. Then $\mathcal{E}(G) \ge \dfrac{2m}{\Delta}$.*

2 Main results

In this section, we state some of new results on the bounds of energy and Sombor index and relationship between them.

Theorem 2.1. *Let G be a connected graph of order n. Then $\mathcal{E}(G) \le \dfrac{SO(G)}{\delta}$.*

Theorem 2.2. *[1] If G is a connected graph of order n which is not $P_n (n \le 8)$, then $\mathcal{E}(G) \le \dfrac{SO(G)}{2}$.*

Theorem 2.3. *For any n $(n \ne 3, 4)$, $\mathcal{E}(P_n) \le \sqrt{2}(n-1)$.*

In [8] was proven that, for any bipartite graph, $\mathcal{E}(G) \le \sqrt{\dfrac{2}{\delta^3}} SO(G)$. In following theorem, we prove that for every graph of order n this inequality holds.

Theorem 2.4. *Let G be a graph. Then $\mathcal{E}(G) \le \sqrt{\dfrac{2}{\delta^3}} SO(G)$. If equality holds, then G is regular.*

In [10] it was demonstrated that for a connected graph of order $n \ge 2$ and size m, $SO(G) \ge \sqrt{2} m\delta$. Now, we give the upper bound for the Sombor index of G and show that $SO(G) \le \sqrt{2} m\Delta$. Therefore,

$$\sqrt{2} m\delta \le SO(G) \le \sqrt{2} m\Delta.$$

Theorem 2.5. *Let G be a graph of size m. Then $SO(G) \le \sqrt{2} m\Delta$.*

In [5] it was shown that for Δ-regular graph G, $SO(G) \le \mathcal{E}(G)\Delta^2$. Also, in [9] it was proven that for Δ-regular graph G, $SO(G) \le \dfrac{\mathcal{E}(G)\Delta^2}{\sqrt{2}}$. Here, we extend this theorem to all graphs.

Theorem 2.6. *Let G be a graph. Then, $SO(G) \le \dfrac{\mathcal{E}(G)\Delta^2}{\sqrt{2}}$.*

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The m-bipartite Ramsey number $BR_m(K_{2,2}, K_{5,5})$

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Abstract

The bipartite Ramsey number $BR(H_1, H_2, \ldots, H_k)$ is the smallest positive integer b, such that each k-decomposition of $E(K_{b,b})$ contains H_i in the i-th class for some $i, 1 \leq i \leq k$. As another view of bipartite Ramsey numbers, for the given two bipartite graphs H_1 and H_2 and a positive integer m, the m-bipartite Ramsey number $BR_m(H_1, H_2)$, is defined as the least integer n, such that any subgraph of $K_{m,n}$ say H, results in $H_1 \subseteq H$ or $H_2 \subseteq \overline{H}$. The size of $BR_m(K_{2,2}, K_{3,3})$, $BR_m(K_{2,2}, K_{4,4})$ for each m, and the size of $BR_m(K_{3,3}, K_{3,3})$ for some m, have been determined in several papers up to now. Also, it is shown that $BR(K_{2,2}, K_{5,5}) = 17$. In this article, we compute the size of $BR_m(K_{2,2}, K_{5,5})$ for some $m \geq 2$.

1 Introduction

In his 1930 on formal logic, F. Ramsey proved that if the t-combinations of an infinite class Γ are colored by d distinct colors, then there exists a subclass $\mathbb{F} \subseteq \Gamma$ so that all of the t-combinations of $\mathbb{F}$ have the same color. For $t = 2$, this is equivalent to saying that an infinite complete graph whose edges are colored in d colors contains an infinite monochromatic complete subgraph. For given two graphs G and H the Ramsey number $R(G, H)$ is the minimum order of a complete graph such that any 2-coloring of the edges must result in either a copy of graph G in the first color or a copy of graph H in the second color. All such Ramsey numbers $R(G, H)$ exist as well. Also, it is shown that $R(G, H) \leq R(K_m, K_n)$ where $|G| = m$ and $|H| = n$.

Beineke and Schwenk, introduced the bipartite version of Ramsey numbers [**?**]. For given bipartite graphs $H_1, H_2, \ldots H_k$, the bipartite Ramsey number $BR(H_1, H_2, \ldots, H_k)$ is the smallest positive integer b, such that each k-decomposition of $E(K_{b,b})$, contains H_i in the i-th class for some $i, 1 \leq i \leq k$. Assume that H_1 and H_2 are two bipartite graphs. For each $m \geq 1$, the m-bipartite Ramsey number $BR_m(H_1, H_2)$, is defined as the least integer n, such that any subgraph of $K_{m,n}$ say H, results in $H_1 \subseteq H$ or $H_2 \subseteq \overline{H}$. The size of $BR_m(H_1, H_2)$ where $H_1 \in \{K_{2,2}, K_{3,3}\}$ and $H_2 \in \{K_{3,3}, K_{4,4}\}$, have been determined in some previous articles. In particular:

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keywords: Ramsey numbers, Bipartite Ramsey numbers, m-bipartite Ramsey number.

Theorem 1.1. *[1, 4] For each positive integer $m \geq 2$, we have:*

$$BR_m(K_{2,2}, K_{3,3}) = \begin{cases} \textit{does not exist}, & \textit{where} \quad m = 2, 3, \\ 15 & \textit{where} \quad m = 4, \\ 12 & \textit{where} \quad m = 5, 6, \\ 9 & \textit{where} \quad m = 7, 8. \end{cases}$$

Theorem 1.2. *[2, 3] For each positive integer $m \geq 2$, we have:*

$$BR_m(K_{3,3}, K_{3,3}) = \begin{cases} \textit{does not exist}, & \textit{where} \quad m = 2, 3, 4, \\ 41 & \textit{where} \quad m = 5, 6, \\ 29 & \textit{where} \quad m = 7, 8. \end{cases}$$

Theorem 1.3. *[5] For each positive integer $m \geq 2$, we have:*

$$BR_m(K_{2,2}, K_{4,4}) = \begin{cases} \textit{does not exist}, & \textit{where} \quad m = 2, 3, 4, \\ 26 & \textit{where} \quad m = 5, \\ 22 & \textit{where} \quad m = 6, 7, \\ 16 & \textit{where} \quad m = 8, \\ 14 & \textit{where} \quad m \in \{9, 10 \dots, 13\}. \end{cases}$$

In this paper, we compute the exact value of $BR_m(K_{2,2}, K_{5,5})$ for some $m \geq 2$ as follows.

Theorem 1.4. *[Main results] For each $m \in \{1, 2, \dots, 8\}$, we have:*

$$BR_m(K_{2,2}, K_{5,5}) = \begin{cases} \textit{does not exist}, & \textit{where} \quad m = 2, 3, 4, 5, \\ 40 & \textit{where} \quad m = 6, \\ 30 & \textit{where} \quad m = 7, 8. \end{cases}$$

2 Preparations

Assume that $G[V, V']$ (or simply $[V, V']$), is a bipartite graph with bipartition sets V and V'. Let $E(G[W, W'])$, denotes the edge set of $G[W, W']$. We use $\Delta(G_V)$ and $\Delta(G_{V'})$ to denote the maximum degree of vertices in part V and V' of G, respectively. The degree of a vertex $v \in V(G)$, is denoted by $\deg_G(v)$. For each $v \in V(V')$, $N_G(v) = \{u \in V'(V), \quad vu \in E(G)\}$. For given graphs G, H, and F, we say G is 2-colorable to (H, F), if there is a subgraph G' of G, where $H \nsubseteq G'$ and $F \nsubseteq \overline{G'}$. We use $G \to (H, F)$, to show that G is 2-colorable to (H, F). To simplify, we use $[n]$ instead of $\{1, \dots, n\}$.

Alex F. Collins et al., have proven the following theorem [6].

Theorem 2.1. *[6] $BR(K_{2,2}, K_{5,5}) = 17$.*

Lemma 2.2. *Suppose that $(X = \{x_1, \dots, x_m\}, Y = \{y_1, \dots, y_n\})$, where $m \geq 6$ and $n \geq 10$ are the partition sets of $K = K_{m,n}$. Let G is a subgraph of $K_{m,n}$. If $\Delta(G_X) \geq 10$, then either $K_{2,2} \subseteq G$ or $K_{5,5} \subseteq \overline{G}$.*

Proof. Without loss of generality (W.l.g), let $\Delta(G_X) = 10$ and $N_G(x) = Y'$, where $|Y'| = 10$ and $K_{2,2} \nsubseteq G$. Therefore, $|N_G(x') \cap Y'| \leq 1$ for each $x' \in X \setminus \{x\}$. Since $|X| \geq 6$ and $|Y'| = 10$, one can check that $K_{5,5} \subseteq \overline{G}[X \setminus \{x\}, Y']$. □

3 Proof of the main results

To prove Theorem 1.4, we need the following theorems.

Theorem 3.1. *For each $m \in \{2,3,4,5\}$, the number $BR_m(K_{2,2}, K_{5,5})$ does not exist.*

Proof. Suppose that $m \in \{2,3,4,5\}$. For an arbitrary integer $t \geq 5$, set $K = K_{m,t}$ and let G be a subgraph of K, such that $G \cong K_{1,t}$. Therefore, we have $\overline{G} \subseteq K_{m-1,t}$. Hence, neither $K_{2,2} \subseteq G$ nor $K_{5,5} \subseteq \overline{G}$. Which means that for each $m \in \{2,3,4,5\}$, the number $BR_m(K_{2,2}, K_{5,5})$ does not exist.

□

In the following theorem, we compute the size of $BR_m(K_{2,2}, K_{5,5})$ for $m = 6$.

Theorem 3.2. $BR_6(K_{2,2}, K_{5,5}) = 40$.

If $\Delta = 8$, then $K_{5,5} \subseteq \overline{G}$.
If either $|N_G(x_i) \cap Y_1| = 0$ or $N_G(x_i) \cap Y_1 = N_G(x_j) \cap Y_1$ for some $i, j \in \{2, \ldots, 6\}$, then $K_{5,5} \subseteq \overline{G}$.

Theorem 3.3. $BR_7(K_{2,2}, K_{5,5}) = BR_8(K_{2,2}, K_{5,5}) = 30$.

Proof of Theorem 1.4. By combining Theorems 3.1, 3.2, and 3.3, we conclude that the proof of Theorem 1.4 is complete. □

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Some results in list $\mathcal{G}$-free colouring of graphs

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Abstract

Let $\mathcal{G}$ is a collection of some graphs and H be a simple graph. The $\mathcal{G}$-free chromatic number $\chi_{\mathcal{G}}(H)$ of H is defined as the minimum number of subsets in a partition of the $V(H)$ such that each subset induces an $\mathcal{G}$-free subgraph, i.e. contains no copy of any member of $\mathcal{G}$. The list $\mathcal{G}$-free $\chi_{\mathcal{G}}^{L}(H)$ of H is the minimum k such that there is an $\mathcal{G}$-free L-coloring for any list assignment L of $V(H)$ which $|L(v)| \geq k$. A graph H is said to be k-$\mathcal{G}$-free-choosable if there exists an L-coloring for any list-assignment L satisfying $|L(v)| \geq k$ for each $v \in V(H)$, and $H[V_i]$ be $\mathcal{G}$-free for each $i = 1, 2, \dots, L$. So $\chi_{\mathcal{G}}(H) \leq \chi_{\mathcal{G}}^{L}(H)$ for any graph H. A particular state, when $\mathcal{G}$ be a ollection of all caycle, $\chi_{\mathcal{G}}(H)$ is said the vertex arboricity of H and shown by $\alpha(H)$. We abtion $\chi_{\mathcal{G}}(H \oplus K_n) = \chi_{\mathcal{G}}^{L}(H \oplus K_n)$, where $\mathcal{G}$ is a collection of graphs with minimum degree δ, H is a fixed graph and n is sufficiently large. Also we show that $\chi_{\mathcal{G}}(H) = \chi_{\mathcal{G}}^{L}(H)$ for some graph H and some family $\mathcal{G}$, and for each H, and H' we prove that $\chi_{\mathcal{G}}^{L}(H \oplus H') \leq \chi_{\mathcal{G}}^{L}(H) + \chi_{\mathcal{G}}^{L}(H')$.

1 Introduction

For given graph H, and each vertex v of H, let $L(v)$ be a set assigned to v, called a color list. An L-coloring of H is a vertex-coloring c such that:

- For any $v \in V(H)$, $c(v) \in L(v)$.
- $c(v) \neq c(v')$ for each $vv' \in E(H)$.

If there exists an L-coloring of H, then H is called L-colorable. A graph H is said to be k-choosable if there exists L-coloring for any list-assignment L satisfying $|L(v)| \geq k$ for each $v \in V(H)$. The choice number $\chi_L(H)$ of H is the minimum integer k such that H is k-choosable. Note that $\chi(H) \leq \chi_L(H)$ for any graph H, however, equality does not necessarily hold. The following particular case has been proved by Ohba [1]:

Theorem 1.1. *[1] If $|V(H)| \leq \chi(H) + \sqrt{2\chi(H)}$, then:*

$$\chi_L(H) = \chi(H).$$

Theorem 1.2. [1] *If $|V(H)| + |V(G)| \leq \chi(H) + \chi(G) + \sqrt{2(\chi(H) + \chi(G))}$, then:*

$$\chi_L(H \oplus G) = \chi(H) + \chi(G).$$

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subjclass[2010]: 05C15
keywords: $\mathcal{G}$-free-choosable, Conditional Coloring, Vertex Arboricity, L-$\mathcal{G}$-free-colorable.

1.1 Conditional Coloring

For a given graph H and a graphical property P, the conditional chromatic number $\chi(H,P)$ of H, is the smallest number k, such that $V(H)$ can be decomposed into sets $V_1, V_2 \dots, V_k$, in which $H[V_i]$ satisfies the property P, for each $1 \leq i \leq k$. This extension of graph coloring was stated by Harary in 1985 [**?**]. A particular state, when P is the property of being acyclic, $\chi(H,P)$ is said the vertex arboricity of H. The vertex arboricity of H is shown $\alpha(H)$ and is defined as the last number of subsets in a partition of the vertex set of H, so that any subset induces an acyclic subgraph. Let $\mathcal{G}$ is a collection of some graphs say $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$. When P is the property that each color class contains no copy of G_i for each $i \in \{1, 2, \dots, n\}$, we write $\chi_{\mathcal{G}}(H)$ instead of $\chi(G,P)$, which is called the $\mathcal{G}$-free chromatic number. Due to this, we say a graph H has a k-$\mathcal{G}$-free coloring if there exists a map $g : V(H) \longrightarrow \{1, \dots, k\}$, such that each of the color classes of g be $\mathcal{G}$-free, see [5].

We use $\alpha_L(H)$ to denote the list vertex arboricity of H, which is the least integer k, such that there exists an acyclic L-coloring for each list assignment L of H, in which $k \leq |L(v)|$. So, $\alpha(H) \leq \alpha_L(H)$ for any graph H, see [3, 4]. It has been proved that:

Theorem 1.3. [2] *If $|V(H)| \leq 2\alpha(H) + \sqrt{2\alpha(H)} - 1$, then $\alpha_L(H) = \alpha(H)$.*

Assume that each vertex $v \in V(H)$ is assigned a set $L(V)$ of colors, told a color list. Set $c(L) = \{c(v) : v \in V(H)\}$. An L-coloring c is called $\mathcal{G}$-free, such that:

- $c(v) \in L(v)$ for each $v \in V(H)$.
- $H[V_i]$ is $\mathcal{G}$-free for each $i = 1, 2, \dots, L$.

If there exists an L-coloring of H, then H is said to be L-$\mathcal{G}$-free-colorable. A graph H is said k-$\mathcal{G}$-free-choosable if there exists an L-coloring for any list-assignment L satisfying $|L(V)| \geq k$ for each $v \in V(H)$, and $H[V_i]$ be $\mathcal{G}$-free for each $i = 1, 2, \dots, L$. For given two graphs H and , the $\chi^L_{\mathcal{G}}(H)$ of H is the minimum integer k, if H be k-$\mathcal{G}$-free-choosable where $\mathcal{G}$ is a collection of some graphs say $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$. When $\mathcal{G} = \{G\}$ we use $\chi_G(H)$ instead to $\chi_{\mathcal{G}}(H)$.

2 Main results

In this article, we prove the subsequent results.

Theorem 2.1. *Assume that $\mathcal{G}$ be a family of graph with minimum degre δ. Also let H and H' are two graphs, where, H is a k-$\mathcal{G}$-free choosable, and H' is a k'-$\mathcal{G}$-free choosable. Suppose that S and S' be the maximum subsets of $V(H)$ and $V(H')$, respectively, such that $H[S]$ and $H'[S']$ are $\mathcal{G}$-free. In this case, if either $(|S'|-1)(|V(H)|+|S'|) \leq |S'|\delta(k+1)$ or $(|S|-1)(|V(H')|+|S|) \leq |S|\delta(k'+1)$, then $H \oplus H'$ is a $(k+k')$-$\mathcal{G}$-free-choosable, that is, $\chi^L_{\mathcal{G}}(H \oplus H') \leq \chi^L_{\mathcal{G}}(H) + \chi^L_{\mathcal{G}}(H') = k + k'$.*

Theorem 2.2. *Let $\mathcal{G}$ is a collection of some graphs say $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$ where for each i we have $\delta(G_i) = \delta$. Also let H be a graph. If $|V(H)| \leq \delta\chi_{\mathcal{G}}(H) + \sqrt{\delta\chi_{\mathcal{G}}(H)} - (\delta - 1)$, then $\chi^L_{\mathcal{G}}(H) = \chi_{\mathcal{G}}(H)$.*

Theorem 2.3. *For each two connected graphs H and G, we have:*

$$\chi^L_{_G}(H) \leq \lceil \frac{\Delta(H)}{\delta(G)} \rceil + 1.$$

Theorem 2.4. *Assume that $\mathcal{G}$ is a collection of all graphs with minimum degre d. For each arbitrary graph H, there exists a non-negative integer n', such that for each $n \geq n'$ we have $\chi_{\mathcal{G}}(H \oplus K_n) = \chi^L_{\mathcal{G}}(H \oplus K_n)$.*

3 Proof of the Main results

Before proving the main theorems, we need some basic results. Suppose that H and G are two graphs, and let L be a list-assignment color to $V(H)$. Assume that $S = \{v_1, v_2, \dots, v_m\} \subseteq V(H)$. Set $L(S) = \cup_{i=1}^{i=m} L(v_i)$. Now, we have the following lemma.

Lemma 3.1. *Let H and G be two graphs with $\delta(G) = \delta$. If H is not L-G-free colorable, then there exists a subset $S \subseteq V(H)$, such that $|S| > \delta|L(S)|$.*

To prove the following lemma, we use Ohbas notion.

Lemma 3.2. *Suppose that H, H', and G are three graphs, where $\delta(G) = \delta$, $|V(H')| = n$, and H' is G-free, i.e $G \nsubseteq H'$. If H be k-G-free choosable, and $(n-1)(|V(H)|+n) \leq n\delta(k+1)$, then $H \oplus H'$ is $(k+1)$-G-free choosable, that is $\chi^L_G(H \oplus H') \leq \chi^L_G(H) + 1 = k + 1$.*

With an argument similar to the proof of the Lemma 3.3, it is easy to check the correctness of the following lemma.

Lemma 3.3. *Let $\mathcal{G}$ is a collection of some graphs with minimum degre δ. Suppose that H, H' are two graphs, where $|V(H')| = n$, and H' is $\mathcal{G}$-free, i.e $G \nsubseteq H'$ for each $G \in \mathcal{G}$. If H be $\chi^L_{\mathcal{G}}(H)$-$\mathcal{G}$-free choosable, and $(n-1)(|V(H)|+n) \leq n\delta(\chi^L_{\mathcal{G}}(H)+1)$, then $H \oplus H'$ is $(\chi^L_{\mathcal{G}}(H)+1)$-$\mathcal{G}$-free choosable, that is $\chi^L_{\mathcal{G}}(H \oplus H') \leq \chi^L_{\mathcal{G}}(H) + 1$.*

By using Lemma 3.3, we can prove the first main result, namely Theorem 2.1.

Lemma 3.4. *Let $\mathcal{G}$ is a collection of some graphs with minimum degre δ. Also suppose that H be a graph with $|V(H)| = n$. Then:*

$$\chi^L_{\mathcal{G}}(H) \leq \lceil \frac{n}{\delta} \rceil$$

Lemma 3.5. *Suppose that H and G are two graphs, where $\delta(G) = \delta$. Assume that S be a maximum G-free subset of H. Also let $|S| = n_1$. If $(n_1 - 1)|V(H)| \leq n_1\delta\chi_G(H)$. Then:*

$$\chi^L_G(H) = \chi_G(H).$$

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The annihilator graph of the power set ring of a set

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Abstract

Let R be a commutative ring. The annihilator graph of R denoted by $AG(R)$ is a graph whose vertices are the nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $\mathrm{Ann}(x) \cup \mathrm{Ann}(y) \subset \mathrm{Ann}(xy)$. In literatures, it is shown that $\mathrm{diam}(AG(R)) \leq 2$ and $\mathrm{gr}(AG(R)) \leq 4$ provided that it has a cycle. In this paper, among other things we introduce a special ring for which the diameter and the girth of the annihilator graph are exactly equal to two and three, respectively.

1 Introduction

In recent decades, graph theoretical tools are used extensively to study rings structures. Therefore, the study of graphs associated with rings has became one of the active area in this field. There are a lot of papers, which apply combinatorial methods to obtain algebraic results in ring theory, see [2]. Let R be a commutative ring with nonzero identity. The annihilator graph of R introduced in [2] and studied in some literatures, for example, see [1]. This graph denoted by $AG(R)$ and its vertices are the nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $\mathrm{Ann}(x) \cup \mathrm{Ann}(y) \subset \mathrm{Ann}(xy)$, where $\mathrm{Ann}(x) = \{r \in R : rx = 0\}$. Let X be a set and let $\mathcal{P}(X)$ denote the power set of X. In this paper, we study the annihilator graph of the ring $(\mathcal{P}(X), \Delta, \cap)$ with the vertex set $Z(\mathcal{P}(X))^* = \mathcal{P}(X) \setminus \{\varnothing, X\}$ and two distinct vertices A and B are adjacent if and only if $\mathrm{Ann}(A) \cup \mathrm{Ann}(B) \subset \mathrm{Ann}(A \cap B)$. We show that for $|X| \geq 3$, $\mathrm{diam}(AG(\mathcal{P}(X))) = 2$ and $\mathrm{gr}(AG(\mathcal{P}(X))) = 3$. Moreover, we determine the degree of all vertices also, we determine the domination, the clique and the chromatic number of this graph.

Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ and $E(G)$ are called vertex set and edge set of G, respectively. A clique of G is a complete subgraph of G and the number of vertices in a largest clique of G denoted by $\omega(G)$, is called the clique number of G. The domination number, denoted by $\gamma(G)$ is the size of a smallest dominating set of G and the chromatic number of G denoted by $\chi(G)$, is the minimal number of colors, which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. For notations and terminologies not given in this paper, the reader is referred to [3].

*Speaker

subjclass[2000]: 13A70, 05C25

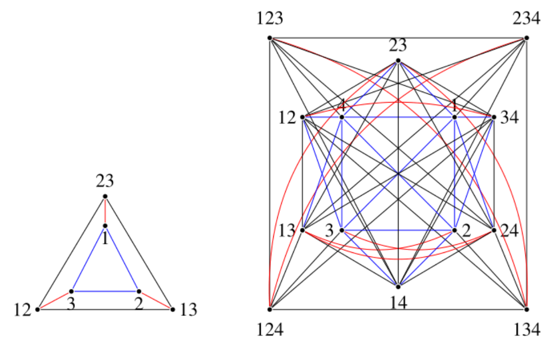
keywords: Annihilator graph, Domination number, Chromatic number

2 The annihilator graph of the ring $\mathcal{P}(X)$

Let X be a set and $\mathcal{P}(X)$ denote the power set of X, $\mathcal{P}(X) = \{A : A \subseteq X\}$. It is easy to see that $(\mathcal{P}(X), \Delta, \cap)$ is a commutative ring with identity, where by $A\Delta B$ we mean the symmetric difference between two sets A and B. In this section we study the annihilator graph of the ring $\mathcal{P}(X)$. For $A \in \mathcal{P}(X)$ it is easy to see that $\mathrm{Ann}(A) = \{B \in \mathcal{P}(X) : A \cap B = \varnothing\} = \mathcal{P}(A')$.

Lemma 2.1. *Let X be a set and $\mathcal{P}(X)$ be the power set of X. Then two distinct vertices A, B in $AG(\mathcal{P}(X))$ are adjacent if and only if $A \notin B$ and $B \notin A$.*

Below are the annihilator graphs for $|X| = 3$ and $|X| = 4$.



Theorem 2.2. *Let X be a set and $\mathcal{P}(X)$ be the power set of X. Then $AG(\mathcal{P}(X))$ is connected graph and $\mathrm{diam}(AG(\mathcal{P}(X))) \leq 2$. Moreover, if $AG(\mathcal{P}(X))$ contains a cycle, then $\mathrm{gr}(AG(\mathcal{P}(X))) = 3$.*

We next determine when $AG(\mathcal{P}(X))$ has a vertex adjacent to every other vertex.

Theorem 2.3. *Let X be a set and $\mathcal{P}(X)$ be the power set of X. Then $AG(\mathcal{P}(X))$ has a vertex adjacent to all vertices if and only if $|X| = 2$. In particular, $AG(\mathcal{P}(X))$ is a complete graph if and only if $|X| = 2$.*

Theorem 2.4. *Let X be a finite set, $|X| = n \geq 1$ and $\mathcal{P}(X)$ be the power set of X. If A is a vertex of $AG(\mathcal{P}(X))$ with k or $n-k$ elements, then $\deg(A) = 2^n - 2^{n-k} - 2^k + 1$.*

In the following we determine the domination, clique and chromatic number of $AG(\mathcal{P}(X))$.

Theorem 2.5. *Let X be a set with more that two elements and $\mathcal{P}(X)$ be the power set of X. Then the domination number of $AG(\mathcal{P}(X))$, $\gamma(AG(\mathcal{P}(X))) = 2$.*

Theorem 2.6. *Let X be a finite set, $|X| = n \geq 1$ and $\mathcal{P}(X)$ be the power set of X. Then $\{A \subseteq X \ : \ |A| = k, 1 \leq k < n\}$ is a clique of $AG(\mathcal{P}(X))$ and $\omega(AG(\mathcal{P}(X))) = \binom{n}{k}$, where $k = n/2$ or $k = (n+1)/2$.*

Proof. It is easy to see that $\Omega_k = \{A \subseteq X \ : \ |A| = k\}$ is a clique of $AG(\mathcal{P}(X))$ for all $1 \leq k < n$. Moreover, if $B \subseteq X$ and $|B| \neq k$, then $\Omega_k \cup \{B\}$ is not a clique since either $B \subseteq A$ or $A \subseteq B$, for some $A \in \Omega_k$, thus A is not adjacent to B and $\Omega_k \cup \{B\}$ is not a clique.

Now, assume that Ω is a clique of $AG(\mathcal{P}(X))$ and $k = \max\{|\Omega \cap \Omega_t| : 1 \leq t < n\}$. Thus $\Omega = \Omega'_k \cup \{A, B, C, \cdots\}$, where $\Omega'_k = \Omega \cap \Omega_k$. We show that $|\Omega| \leq |\Omega_k|$. For $A \in \Omega$ we have two cases: (i) if $|A| > k$, then there is no any subset of A with k elements in Ω'_k; (ii) if $|A| < k$, then there is no any subset of X with k elements in Ω'_k which contains A. The number of subsets of A with k elements is $\binom{|A|}{k}$ and the number of subsets of X with k elements which contains A is $\binom{n-|A|}{k}$. In any cases the number of these subsets of X is more than one. Thus $|\Omega| \leq |\Omega_k|$ the result follows. $\square$

Theorem 2.7. *Let X be a finite set, $|X| = n \geq 1$ and $\mathcal{P}(X)$ be the power set of X. Then the chromatic number of $AG(\mathcal{P}(X))$ is $\chi(AG(\mathcal{P}(X))) = \binom{n}{k}$, where $k = n/2$ or $k = (n+1)/2$.*

Proof. In view of Theorem 2.6, $\Omega_k = \{A \subseteq X \ : \ |A| = k, k = n/2 \text{ or } k = (n+1)/2\}$ is a maximal clique of $AG(\mathcal{P}(X))$ so $\binom{n}{k} \leq \chi(AG(\mathcal{P}(X)))$, where $k = n/2$ or $k = (n+1)/2$. Assume that A is an arbitrary vertex of $AG(\mathcal{P}(X))$. Since Ω_k is a maximal clique so there is a vertex B in Ω_k such that A and B are not adjacent. Hence, A and B have the same color. Now, assume that C is a vertex of $AG(\mathcal{P}(X))$ which is adjacent to A. We have to show that A and C have different colors. We may assume that $C \notin \Omega_k$. Thus either $n/2 < |C|$ or $|C| < n/2$. In the first case there exist $\binom{|C|}{n/2} \geq 2$ subsets of C in the clique Ω_k which C is not adjacent to them and therefore C can be have the same color as one of them other than B. So A and C have different colors. If $|C| < n/2$, then there exist $\binom{n-|C|}{n/2} \geq 2$ subsets of X in the clique Ω_k contains C so C is not adjacent to them and therefore can be have the same color as one of them other than B. So A and C have different colors. Hence, in each cases A and C have different colors. Therefore, $\chi(AG(\mathcal{P}(X))) \leq |\Omega_k|$, where $k = n/2$ or $k = (n+1)/2$. □

Corollary 2.8. *Let X be a finite set and $\mathcal{P}(X)$ be the power set of X. Then $AG(\mathcal{P}(X))$ is a weakly perfect graph.*

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D-Graphs

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Abstract

One of the famous problem in elementary combinatorics is: count the number of ways of distributing n identical objects into k distinct labeled boxes. In this study, we aim to present a novel interconnection topology called the Diophantine Cube. In spite of its symmetric and relatively sparse interconnections, the Diophantine Cube has shown to posess attractive structures. Since it can be embedded as a subgraph in Hamming Graphs, the Diophantine cube may find applications in fault-tolerant computing.

1 Introduction

There are a large number of graphs whose principles of structure are designed based on strings of length k. One of these interesting problems is the famous (generalized) Tower of Hanoi puzzle [1]. In its general form the puzzle consists of $p \geq 3$ vertical pegs numbered $1, 2, \ldots, p$ and n $(n, p \in \mathbb{N})$ discs of different size numbered $1, 2, \ldots, n$, where the discs are ordered by size , disc 1 being the smallest one. A *state*, that is, a distribution of discs on pegs, is called *regular*. Starting from a *perfect* state, a regular state where all discs are on one peg . A *legal move* is a transfer of topmost disc of a peg to another peg such that no disc is moved onto smaller one.The aim of the game is to transfer all discs from one perfect state to another in the minimum number of legal moves.

A regular state can be represented by a unique $n-$string $p_1p_2\ldots p_n$, here p_j is the peg where disc j, is laying. Conversely, any such n-string determines a unique regular state. To see this , we first recall that the coordinates define the discs on each peg. Scince the discs on each peg are ordered by their sizes in a regular state. We conclude that there are p^n regular state. We consider the set of all regular state n-string as the vertex set of Hanoi Graph H_p^n and two vertices are adjacent if and only if one can obtained from each other by a legal move of a single disc [2].

Now, in the Tower of Hanoi problem, if we consider all discs are the same diameter and to be identical, and suppose we consider the pegs as labbled different boxes. In this article, we introduce a graph whose set of vertices are codes of length k that are obtained from an important combinatorial structure. Graphs whose vertices are strings, each of which represents a distribution of n identical objects in k distinct boxes. Since, we know that every way of distribution of a n similar objects in k distinct boxes is equivalent to a solution of the Diophantine equation $x_1 + x_2 + \ldots + x_k = n$. For this purpose, we call the obtained graphs a distribution graph or Diophantine graph or simply a $D - Graph$.

*Speaker

subjclass[2010]: 05C12, 05C78, 11B37

keywords: Diophantine equations; increasing strings; coloring; distance.

2 Diophantine Graphs D_k^n

A convenient and direct representation for the Distribution Problem is graph representation. In a Distribution Problem every possible state of the Problem is represented by a vertex. Two vertices are adjacent in the Diophantine Graph if their corresponding states differ by one move. In this section we define the *Diophantine graph* and investigate some basic parameter of it. But, before this we need the following definitions. Recall that the Hamming distance between two binary strings α and β is the number $H(\alpha,\beta)$ of bits, where α and β differ [2]. Now, we generalize this concept to Diophantine codes.

Definition 2.1. *Let $n,k \geq 1$ are positive integers.The Diophantine graph D_k^n of kind (n,k) is the graph$D_k^n = (V_k^n, E_k^n)$, where $V_k^n = \{a_1a_2\dots a_n : 1 \leq a_1 \leq a_2 \leq \dots \leq a_n \leq k\}$,that is the set of all Diophantine code of the kind (n,k) and $(\alpha,\beta) \in E_k^n$ if and only if $H(\alpha,\beta) = 1$.*

Example 2.2. *Let $k = 2$ and $n = 3$, then $V_2^3 = \{111, 112, 122, 222\}$ and $(3,2)-$Diophantine Graph D_2^3 is isomorphic to P_4.* Figure 1(b).

For each $n \geq 1$, one can simply show that, $D_2^n \simeq P_{n+1}$.
Now, we want to draw the Diophantine graph corresponding to the non-negative integer solutions of the

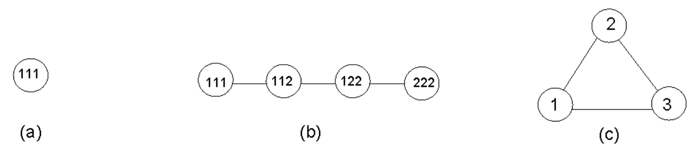

Figure 1: Diophantine Graphs

equation $x_1+x_2+x_3 = 1$. We want to show each solution of the equation by 1-string over the set $\{1,2,3\}$.The solutions of this equation are $x_1 = 1, x_2 = x_3 = 0$, which correspond to the 1-string $a_1 = 1$; or the solution $x_2 = 1, x_1 = x_3 = 0$ where correspond to the 1-string $a_1 = 2$; and the last solution $x_3 = 1, x_1 = x_2 = 0$, which correspond to the 1-string $a_1 = 3$. *Figure 1(c).*
Therefore, we have three distinct sequences of length 1 which, each pair is different in one component and their corresponding vertices in Diophantine graph are adjacent. It is a simple problem in the literature of combinatorics (cf. [3]) to see that, each distribution of n identical objects in k labeled distinct boxes is corresponding to a solution of the Diophantine equation $x_1 + x_2 + \dots + x_k = n$, and hence corresponding to an increasing $n-$string over the set $\{1,2,\dots,k\}$. So, we have the following lemma:

Lemma 2.3. $|V_k^n| = \binom{n+k-1}{k-1}$.

Now, we find some structural properties of the Diophantine graphs. Since each vertex of the Diophantine graph corresponds to a solution of the Diophantine linear equation $x_1 + x_2 + \dots + x_k = n$.

Theorem 2.4. *Let $n \geq 1$ and $k \geq 1$ be positive integers. Let $\alpha = a_1a_2\dots a_n$ is an arbitrary vertex of the Diophantine graph D_k^n, then*

$$deg_{D_k^n}(\alpha) = (a_n - a_1) + (k-1)$$

.

In theorem 2.4 we show that the degree of each vertices $a_1a_2\dots a_n$ of $(n,k)-$Diophantine Graphs, dependents only a_1 and a_n, by this fact and the hand shaking theorem [2], we can find the number of edges of the $(n,k)-$Diophantine Graphs $D_n^{(k)}$.

Theorem 2.5. *Let $q_n^{(k)} = |E_n^{(k)}|$ the number of edges of the Diophantine Graph $D_n^{(k)} = (V_n^{(k)}, E_n^{(k)})$, then*

$$q_n^{(k)} = \begin{cases} \binom{k}{2} & \text{if} \quad n = 1 \\ \frac{(k-1)k(k+1)}{3} & \text{if} \quad n = 2 \\ \frac{1}{2}\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\binom{n+j-2}{j}j + \frac{1}{2}\binom{n+k-1}{k-1}(k-1) & \text{if} \quad n \geq 3 \end{cases}$$

3 Coloring the Diophantine Graphs D_k^n

In this section, we use this labbeling, which is key to coloring the vertices. It is customary to number the boxes 0,1,...,$k-1$.. We are now ready to prove a new result. As, induced subgraph $<\{11\dots 1a_n | 1 \le a_n \le k\}>$ is isomorphic to K_k, thus $\chi(D_k^n) \ge k$. To see that k colors suffice, color the vertex labeled $a_1a_2\dots a_n$ by the sum of its box numbers modulo k. That is

$$f(a_1a_2\dots a_n) = a_1 + a_2 + \dots + a_n \quad mod(k).$$

To check that f is a k-coloring, observe that two vertices of D_k^n are adjacent if and only if they differ in exactly one place. Hence, we have the following theorem:

Theorem 3.1. *Let $k \ge 2$ and $n \ge 1$, then $\chi(D_k^n) = k$.*

.

4 Connectivity

In this section, we want to show that for each $(n,k)-$ Diophanteen graps D_k^n are connected graphs where $n,k \ge 1$ are positive integers.

Theorem 4.1. *Let $n,k \ge 1$ are positive integers. Thus the Diophanteen graph D_k^n is connected.*

Lemma 4.2. *Let A and B are two arbitrary vertices of the graph D_k^n. Thus, there is a corner vertex $\bar{i}$ such that, $d(A,\bar{i}) \ge d(A,B)$.*

Thus, for calculating the eccentricity of a vertex$A$, it is sufficient to only calculate its distance from the Corner vertices. So,

$$ecc(A) = \max\{d(A,\bar{i}) : i = 1,2,\dots,k\}.$$

Since, the corner vertices $\bar{1} = 11\dots 1$ and $\bar{k} = kk\dots k$ are differ on n coordinates, then $diam(D_k^N) = n$. Thus, we have the following lemma.

Lemma 4.3. *For each positive integers $n \ge 1$and $k \ge 1$, the diameter of the Diophanteen graph D_k^n is n.*

By lemma 4.2, for calculating the eccentricity of a vertex$A$, it is sufficient to only calculate its distance from the Corner vertices. Indeed, for each vertex $A \in D_k^n$, we have

$$\begin{aligned} ecc(A) &= \max\{d(A,X) : X \in V(D_k^n)\} \\ &= \max\{d(A,\bar{i}) : i \in \{1,2,\dots,k\}\} \\ &= \max\{n - n_i : i = 1,2,\dots,k\} \end{aligned}$$

where $n_i = |M_i(A)|$. Let A is a vertex of the graph D_k^n, then by lemma 4.2 there is a corner vertex $\bar{i}$, such that $ecc(A) = n - n_i$. If $n_i < \lfloor \frac{n}{k} \rfloor$, then $n - n_i > n - \lfloor \frac{n}{k} \rfloor$. But, eccentericity of the vertex $B = \overbrace{1\dots 1}^{n_1}\overbrace{2\dots 2}^{n_2}\dots\overbrace{k\dots k}^{n_k}$ is $n - \lfloor \frac{n}{k} \rfloor$ where, $n_i \ge \lfloor \frac{n}{k} \rfloor$ for $i = 1,2,\dots,n$. Hence, the vertex A can not be a centeral vertex. Thus, we have the following lemma.

Lemma 4.4. $rad(D_k^n) = n - \lfloor \frac{n}{k} \rfloor$.

Now, by Lemma 4.2 and Lemma 4.4 one can consruct many graphs with given eccentericity.

Theorem 4.5. *Let $n \ge 1$ and $k \ge 1$ are positive integers then, the center $Z(D_k^n)$ is an induced subgraph of the Diophanteen graph D_k^n of order $\binom{r+k-1}{k-1}$ where $r = n - k\lfloor \frac{n}{k} \rfloor$.*

Corollary 4.6. *Let $n \ge 1$ and $k \ge 1$ are positive integers. then,*

a) If $n < k$ then $Z(D_k^n) \cong D_k^n$ that is, the Diophanteen graph is self center.

b) If $k|n$ then, the Diophanteen graph D_k^n is mono center that is $Z(D_k^n) \cong K_1$ with center $V(Z(D_k^n)) = \{\overbrace{1\dots 1}^{m}\overbrace{2\dots 2}^{m}\dots\overbrace{k\dots k}^{m}\}$ where $m = \frac{n}{k}$.

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2-Coupon Coloring of Cubic Graphs Containing 3-Cycle or 4-Cycle

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Abstract

A total dominating set in a graph G is a set S of vertices of G such that every vertex in G is adjacent to a vertex in S. Motivated by a question in [3], which states that:
"Is it true that every connected cubic graph containing a 3-cycle has two vertex disjoint total dominating sets?"
We give a negative answer to this question. Moreover, we prove that if we replace 3-cycle with 4-cycle the answer is affirmative. This implies every connected cubic graph containing a diamond (the complete graph of order 4 minus one edge) as a subgraph can be partitioned into two total dominating sets, a result that was proved in 2017.

Joint Work with: S. Akbari, M. Azimian, A. Fazli Khani, E. Zahiri

1 Introduction

Throughout this talk, all graphs are simple that is with no loop and multiple edges. Let G be a graph. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. A *total dominating set* in a graph G is a set $S \subseteq V(G)$ such that every vertex in G is adjacent to a vertex in S. The study of cubic graphs whose vertex set can be partitioned into two total dominating sets is an attractive topic and many authors have investigated this problem, for instance, see [1], [3] and [6]. A *k-coloring* of a graph G is an assignment of colors from the set $[k] = \{0, \dots, k-1\}$ to the vertices of G. A *k-coupon* coloring of a graph G without isolated vertices is a k-coloring of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. We say that a vertex has the *coupon property* if all k colors appear in the neighborhood of that vertex. The concept of k-coupon coloring of graphs has been studied by several authors, for instance, see [2], [4] and [5].
The *open neighborhood* of a vertex v in G, denoted by $N_G(v)$, is the set of all vertices adjacent to v. For any $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. A *$\{1,2\}$-factor* is a spanning subgraph of G in which each component is either 1-regular or 2-regular. We use C_n and P_n, for the cycle and the path of order n, respectively. Also, C_n is called an *n-cycle*. A vertex v is called a *good vertex* or a *bad*

*Speaker

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vertex, if it has or has not the coupon property, respectively. It is not hard to see that a 2-coupon coloring of a graph is equivalent to the vertex partitioning of a graph into two total dominating sets. In [3] the following interesting question was proposed:
"*Is it true that every connected cubic graph containing a* 3*-cycle has two vertex disjoint total dominating sets?*"
Here, we construct a connected cubic graph of order 60 containing a 3-cycle whose vertex set cannot be partitioned into two total dominating sets. The complete graph of order 4 minus one edge is called a *diamond*. In [3] it is shown that the vertex set of a cubic graph containing a diamond as a subgraph can be partitioned into two total dominating sets. Here, we strengthen this result by showing that the vertex set of every cubic graph containing a 4-cycle can be partitioned into two total dominating sets.

2 Main results

In this section, we state some of new results on the concept of 2-coupon coloring of cubic graphs.
In this talk, we denote the Heawood graph by $\mathcal{H}$. First we prove the following two lemmas.

Lemma 2.1. *In any* 2*-coloring of* $\mathcal{H}$*, there are at least two bad vertices.*

Lemma 2.2. *Let* $e = uv \in E(\mathcal{H})$*. If there is a* 2*-coloring for* $\mathcal{H} - e$ *such that all vertices in* $V(\mathcal{H})\setminus\{u,v\}$ *are good, then* u *and all vertices in* $N_{\mathcal{H}-e}(v)$ *have the same color.*

Using the two previous lemmas, we prove the following theorem.

Theorem 2.3. *There exists a connected cubic graph containing a* 3*-cycle with no* 2*-coupon coloring.*

Now, we have the following corollary.

Corollary 2.4. *There exists a connected cubic graph containing a triangle that has no two vertex disjoint total dominating sets.*

Theorem 2.5. *Let* G *be a connected cubic graph and* $S \subsetneq V(G)$*. If* $G[S]$ *has a* 2*-coupon coloring, and* $G\setminus S$ *has a* $\{1,2\}$*-factor, then* G *has a* 2*-coupon coloring.*

Using Theorem 2.5 we have the following corollary.

Corollary 2.6. *If* G *is a connected cubic graph containing a* 4*-cycle, then* G *has a* 2*-coupon coloring.*

Remark 2.7. *For every positive integer* r $(r \geq 3)$*, not divisible by* 4*, there exists a connected cubic graph that contains an induced* r*-cycle and has no* 2*-coupon coloring.*

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Some Notes on the Energy of Graphs with Self-Loops

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Abstract

Let G be a graph and $S \subseteq V(G)$. The self-loop graph of G_S is a graph obtained by G by attaching a self-loop at each vertex in S. The energy of a self-loop graph is defined as $\mathcal{E}(G_S) = \sum_{i=1}^{n} |\lambda_i(G_S) - \frac{|S|}{n}|$. It was conjectured that for every graph of order at least 2, there exists a subset $S \subseteq V(G)$ such that $\mathcal{E}(G_S) > \mathcal{E}(G)$. In this paper, we prove this conjecture. Also we show that if G is a bipartite graph of odd order, then for every $\varnothing \neq S \subsetneqq V(G)$, we have $\mathcal{E}(G_S) > \mathcal{E}(G)$. It is shown that for every graph G with nullity r, there is $S \subseteq V(G)$ such that $|S| = r$ and the adjacency matrix of G_S is non-singular.

Joint work with: S. Akbari, S. Küçükçifçi

1 Introduction

The *energy* of a graph G of order n is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

The graph obtained from G by attaching a self-loop at each vertex in $S \subseteq V(G)$ is called the *self-loop graph* of G at S and denoted by G_S. Also, we denote $\overline{S}$ as the complement of S. Define $J_S \in M_{|V(G)|}(\mathbb{C})$ such that $(J_S)_{i,j} = 1$ if $i = j$ and $v_i \in S$, and $(J_S)_{i,j} = 0$, otherwise. Thus we have $A(G_S) = J_S + A(G)$. The characteristic polynomial and the eigenvalues of G_S are the characteristic polynomial and the eigenvalues of $A(G_S)$, respectively. Since $A(G_S)$ is a real symmetric matrix, all eigenvalues of $A(G_S)$ are real and they are denoted by $\lambda_1(G_S) \geq \lambda_2(G_S) \geq \cdots \geq \lambda_n(G_S)$. In [2], the energy of G_S of order n is defined as

$$\mathcal{E}(G_S) = \sum_{i=1}^{n} |\lambda_i(G_S) - \frac{|S|}{n}|.$$

In [2], it was conjectured that for every non-empty proper subset S of $V(G)$, we have $\mathcal{E}(G_S) > \mathcal{E}(G)$. This was disproved in [3] by presenting some counterexamples such that $\mathcal{E}(G_S) < \mathcal{E}(G)$. Instead, we prove the following conjecture, which was proposed in [1].

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*Speaker

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keywords: The energy of graphs with self-loops, Graph energy, Combinatorial Nullstellensatz.

Conjecture 1.1. *For every graph G of order at least 2, there exists $S \subseteq V(G)$ such that $\mathcal{E}(G_S) > \mathcal{E}(G)$.*

Also, in [1] it was shown that if G is a bipartite graph and S is an arbitrary set of $V(G)$, then $\mathcal{E}(G_S) \geq \mathcal{E}(G)$. Here, we will prove that if G is a bipartite graph of odd order and $\varnothing \neq S \subsetneq V(G)$, then the inequality is strict.

2 Main results

In this section, we state some of new results on the the energy of graphs with self-loops.

Theorem 2.1. *Let G be a graph. Then the two following statements hold:*
(i) *If $S \subseteq V(G)$, then $\mathcal{E}(G_S) \geq \mathcal{E}(G)$ or $\mathcal{E}(G_{\overline{S}}) \geq \mathcal{E}(G)$.*
(ii) *For every integer r, $0 \leq r \leq n$, there exists $S \subseteq V(G)$ such that $|S| = r$ and $\mathcal{E}(G_S) \geq \mathcal{E}(G)$.*

Theorem 2.2. *For every graph G, if $\varnothing \neq S \subsetneq V(G)$ and the adjacency matrix of the subgraph of G induced by S is singular, then $\mathcal{E}(G_S) > \mathcal{E}(G)$ or $\mathcal{E}(G_{\overline{S}}) > \mathcal{E}(G)$.*

Corollary 2.3. *For every graph G of order at least 2, there exists $S \subseteq V(G)$ such that $\mathcal{E}(G_S) > \mathcal{E}(G)$.*

Theorem 2.4. *Let G be a graph of order n. Then for every integer r, $0 < r < n$, there exists $S \subseteq V(G)$ such that $|S| = r$ and $\mathcal{E}(G_S) > \mathcal{E}(G)$.*

Theorem 2.5. *Let G be a graph with a singular adjacency matrix and $\varnothing \neq S \subsetneq V(G)$, then either $\mathcal{E}(G_S) > \mathcal{E}(G)$ or $\mathcal{E}(G_{\overline{S}}) > \mathcal{E}(G)$.*

Corollary 2.6. *Let G be a bipartite graph of odd order and $\varnothing \neq S \subsetneq V(G)$, then $\mathcal{E}(G_S) > \mathcal{E}(G)$.*

Corollary 2.7. *For every graph G of order at least 2, there exists $S \subseteq V(G)$ such that $\mathcal{E}(G_S) > \mathcal{E}(G)$.*

Theorem 2.8. *For every graph G with $null(G) = r$, there exists $S \subseteq V(G)$ such that $|S| = r$ and $A(G_S)$ is non-singular.*

Corollary 2.9. *Let G be a regular graph. Then for every $S \subseteq V(G)$, G_S is non-hypoenergetic.*

Acknowledgement. This is joint work with Prof. Saieed Akbari and Prof. Selda Küçükçifçi.

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Degree saving group of a simple graph and its characterization

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Abstract

In this paper, we consider the degree saving group of a simple graph G, denoted by $\Gamma_d(G)$. Then, we characterize the group G for which the vertex group is equal to $\Gamma_d(G)$, and the group G for which the total group is isomorph to $\Gamma_d(G)$.

any two of these groups are isomorphism.

1 Introduction

Let $G = \big(V(G), E(G)\big)$ be a simple graph, i.e., finite undirected graphs with no loops and no multiple edges. Each member of $V(G) \cup E(G)$ is called an *element* of G. We say two elements of G are *associated* if they are either adjacent or incident. The neighborhood of a vertex $\nu \in V(G)$, denoted by $N_G(\nu)$, is the set consisting of all vertices of G which are adjacent to ν. The *closed neighborhood* of $\nu \in V(G)$ is defined as $\overline{N}_G(\nu) \coloneqq N_G(\nu) \cup \{\nu\}$. The degree of $\nu \in V(G)$ is denoted by $\deg(\nu)$.

The set of all permutations on $V(G)$ which preserve adjacency under composition forms a group, denoted by $\Gamma_v(G)$ and called the *vertex group* of G. Also, the set of all permutations of elements of G which preserve association forms another group, denoted by $\Gamma_t(G)$ and named the *total group* of G. To study different properties and examples of $\Gamma_v(G)$ and $\Gamma_t(G)$, we can refer to [2] and [1], respectively. In this article, we consider the third type of permutation groups and introduce its interrelations with $\Gamma_v(G)$ and $\Gamma_t(G)$.

2 Main results

Definition 2.1. *The* degree saving group *of G, denoted by $\Gamma_d(G)$, consists all permutations on $V(G)$ with the property that*

$$deg(\nu) = deg\big(\varphi(\nu)\big), \qquad \forall \nu \in V(G),\ \forall \varphi \in \Gamma_d(G).$$

Owing to the definition, it is clear that $\Gamma_v(G)$ is a subgroup of $\Gamma_d(G)$. The next two theorems characterize graphs G for which $\Gamma_d(G) = \Gamma_v(G)$. In the first theorem we consider disconnected graphs.

*Speaker

subjclass[2010]: 05C31, 05C60

keywords: Simple graph, Degree saving group, Vertex group, Total group.

Theorem 2.2. *Let G be a disconnected graph. Then $\Gamma_d(G) = \Gamma_v(G)$ if and only if no two nontrivial components of G have vertices of the same degree and for every component H of G we have $\Gamma_d(H) = \Gamma_v(H)$.*

Sketch of proof: If $\mathcal{G}_d(G) = \mathcal{G}_v(G)$ and there exist two nontrivial components H_1 and H_2 of G with vertices $v_1 \in H_1$ and $v_2 \in H_2$ with $deg(v_1) = deg(v_2)$, then we consider the permutation $\varphi : V(G) \to V(G)$, defined by

$$\varphi(\nu) := \begin{cases} \nu & if \;\; \nu \in V(G) \smallsetminus \{\nu_1, \nu_2\}, \\ \nu_2 & if \;\; \nu = \nu_1, \\ \nu_1 & if \;\; \nu = \nu_2. \end{cases}$$

Clearly $\varphi \in \mathcal{G}_d(G) \smallsetminus \mathcal{G}_v(G)$, and the necessary is proved. For prove the sufficiency, we consider an arbitrary permutation $\mathcal{G}_d(G)$. Since $maps vertices of nontrivial component$ H$to itself, the restriction of$ to $V(H)$ saves adjacency. Hence, $\mathcal{G}_v(G)$, and the proof is complete. □

According to the Theorem 2.2, we must confine ourselves to connected graphs. The proof of the following theorem is longer than can be presented here, and we refrain from stating it.

Theorem 2.3. *Suppose that $H$is a connected graph. Then $\Gamma_d(H) = \Gamma_v(H)$ if and only if*

($\mathcal{A}$) *equidegree vertices of H two of which are disjoint are all mutually disjoint and all have the same neighborhood, and*

($\mathcal{B}$) *equidegree vertices of H two of which are adjacent are all mutually adjacent and all have the same closed neighborhood.*

The following corollary is immediate from Theorem 2.3. Among regular graphs G, the complete graph and its complement are the only graphs for which $\Gamma_d(G) = \Gamma_v(G)$.

Theorem 2.4. *Among trees having 2 or more vertices, the star graph $K_{1,n}$ is the only tree for which $\Gamma_d(K_{1,n}) = \Gamma_v(K_{1,n})$.*

Sketch of proof: Let T be a tree whose diameter is greater than 2 and $\Gamma_d(T) = \Gamma_v(T)$. Then T contains two nonadjacent vertices of degree one, and this contradicts Theorem 2.3. Thus diameter of T is equal to 2, and the result is proved. □

Theorem 2.5. *Suppose that $\Gamma_d(G) = \Gamma_v(G)$ and v is a cut-vertex of G. Then*

$$deg(v') \neq deg(v) \qquad \forall v' \in V(G) \smallsetminus \{v\}.$$

Sketch of proof: Assume to the contrary that there exists a vertex $v' \neq v$ such that $degv' = degv$. Then, Theorem 2.2 concludes that v and v' belong to the same component H of G. Suppose that u and w are two vertices of H adjacent to v which lie in two different components of $H \smallsetminus \{v\}$. If $v' \in \{u, w\}$, that is, if, say, $v' = u$, then $uw \in E(G)$ contradicts the fact that v is a cut vertex of G. Hence, we assume that $v' \notin \{u, w\}$. Then v' is adjacent to both u and w and again we contradict the fact that v is a cut-vertex of G. This completes the proof. □

The next theorem deals with $\Gamma_v(G)$ and $\Gamma_t(G)$.

Theorem 2.6. *([1, Theorem 3]) Let G be a nontrivial graph. Then, $\Gamma_v(G)$ and $\Gamma_t(G)$ are isomorphic if and only if neither a component of G is a complete graph nor a cycle.*

The following result, deals with $\Gamma_d(G)$ and $\Gamma_t(G)$, is immediate from Theorems 2.3 and 2.6.

Theorem 2.7. *Let G be a nontrivial connected graph. Then, $\Gamma_d(G)$ and $\Gamma_t(G)$ are isomorphic if and only if G is not a complete graph and the statements ($\mathcal{A}$) and (B) hold.*

Remark 2.8. *Since the elements of $\Gamma_d(G)$ are permutations on $V(G)$ and the elements of $\Gamma_t(G)$ are permutations on $V(G) \cup E(G)$, they can only be isomorph, but they cannot be exactly equal. But, the elements of $\Gamma_d(G)$ and $\Gamma_v(G)$ are permutations on $V(G)$, and they can be equal.*

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A new method for computation of Frobenius number

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Abstract

We say that a number α has a representation with respect to the numbers $\alpha_1, ..., \alpha_n$, if the non-negative integers $\lambda_1, ..., \lambda_n$ be found so that $\alpha = \lambda_1\alpha_1 + ... + \lambda_n\alpha_n$. The largest natural number that does not have a representation compared to the numbers $\alpha_1, ..., \alpha_n$ is called the Frobenius number and is denoted by the symbol $g(\alpha_1, ..., \alpha_n)$. In this paper, we present a new algorithm to calculate the Frobenius number.

1 Introduction

Let $\alpha_1, ..., \alpha_n$ $(n \geq 2)$ be positive integers with $\gcd(\alpha_1, ..., \alpha_n) = 1$. Finding the largest positive integer N such that the Diophantine equation $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = N$ has no solution in non-negative integers is known as the Frobenius problem. Such the largest positive integer N is called the Frobenius number of $\alpha_1, ..., \alpha_n$. Various results of the Frobenius number have been studied extensively.

The Frobenius Problem is well known as the coin problem that asks for the largest monetary amount that cannot be obtained using only coins in the set of coin denominations which has no common divisor greater than 1. This problem is also referred to as the McNugget number problem introduced by Henri Picciotto. The origin of this problem for $n = 2$ was proposed by Sylvester (1884), and this was solved by Curran Sharp (1884), see [3, 4]. Curran Sharp [3] in 1884 proved that $g(\alpha_1, \alpha_2) = \alpha_1\alpha_2 - \alpha_1 - \alpha_2$.

Let $\alpha_1, ..., \alpha_n$ be positive integers whose greatest divisor is equal to one, in other words

$$\gcd(\alpha_1, ..., \alpha_n) = 1.$$

If $S =< \alpha_1, ..., \alpha_n >$ is the semigroup generated by $\alpha_1, ..., \alpha_n$, then finding $g(S)$ is a problem and therefore finding the bounds for $g(S)$, whenever we have a certain sequence of numbers, [1] is of interest. For example, if S is an arithmetic sequence with relative value d, then we have [2]:

$$g(a, a + d, a + 2d, ..., a + kd) = a\lfloor \frac{a-2}{k} \rfloor + d(a - 1).$$

Fibonacci sequence is a recursive sequence $F_n = F_{n-1} + F_{n-2}$, $n \geq 3$ with $F_1 = F_2 = 1$.

The following lemmas are needed:

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*Speaker

subjclass[2010]: 01B39, 11D04
keywords: Algorithm, Frobenius, Number

Lemma 1.1. *If $i \nmid l$ then $gcd(F_i, F_l) = 1$.*

Lemma 1.2. *For every integer $l \geq i+2$, we have $g(F_i, F_{i+1}, F_l) = g(F_i, F_{i+1})$.*

Assuming $gcd(F_i, F_j, F_l) = 1$ for the triplet $3 \leq i < j < l$, calculating $g(F_i, F_j, F_l)$ has been considered.

Theorem 1.3. [5] *Suppose that $i, k \geq 3$ are integers and $r = \lfloor \frac{F_i - 1}{F_i} \rfloor$. In this case*

$$g(F_i, F_{i+2}, F_{i+k}) = \begin{cases} (F_i - 1)F_{i+2} - F_i(rF_{k-2} + 1); & \text{If } r = 0 \text{ or } r \geq 1 \text{ and} \\ & F_{k-2}F_i < (F_i - rF_k)F_{i+2}, \\ & \\ r(F_k - 1)F_{i+2} - F_i((r-1)F_{k-2} + 1) & \text{otherwise} \end{cases}$$

2 Main results

We start this section with the following easy theorem:

Theorem 2.1. *For the numbers $\alpha_1 < \alpha_2 < ... < \alpha_n$, we have*

$$g(\alpha_1, ..., \alpha_n) \leq g(\alpha_1, ..., \alpha_{n-1}).$$

Using the upper bound of Theorem 2.1, we present a new algorithm for calculating Frobenius numbers. More precisely, since $g(\alpha_1, ..., \alpha_n) \leq g(\alpha_1, \alpha_2) = \alpha_1\alpha_2 - \alpha_1 - \alpha_2$, we compute the number $\alpha_1\alpha_2 - \alpha_1 - \alpha_2$ and using one sub-algorithm examine the natural numbers less than $g(\alpha_1, \alpha_2)$ are representable respect to $\alpha_1, ..., \alpha_n$. Obviously the largest number less than $\alpha_1\alpha_2 - \alpha_1 - \alpha_2$ which dose not have a representation, is the Frobenius number of $\alpha_1, ..., \alpha_n$. This sub-algorithm determines whether a number has a representation or not compared to the desired numbers. After that, we apply another algorithm to compute the Frobenius number of $\alpha_1, ..., \alpha_n$.

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Matching in (3,6)-Fullerene Graphs

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Abstract

A (3,6)-fullerene graph is 3-connected or 2-connected, 3-regular and, planar graph with triangles and hexagon faces. A set of edges of a graph G is named a matching if no two edges of it have a vertex in common. In this paper, we investigate matchings of k edges in (3,6)-fullerene graphs and calculate the number of these matchings.

1 Introduction

A (3,6)-Fullerene graph is a cubic, 2 or 3-connected planar graph which have four triangular faces. (3,6)-Fullerenes can have isolated or adjacent triangles and, on this basis, divided into two categories. (3,6)-fullerenes with isolate triangles are called non-trivial and (3,6)-fullerenes with a pair of adjacent triangles are known as trivial. Every (3,6)-fullerene is determined by an ordered triple (r, s, t) of non-negative integers, where r is the number of layers of hexagons, s is the number of radial edges in each layer, and t is the twist. For a systematic introduction to fullerene graphs, we refer the reader to [1, 2].

A set of edges of graph G is a matching if no two edges of it have a vertex in common. A matching in a graph G is perfect if each vertex of G is incident with an edge from said matching. In 1891, Petersen established that every 3-regular graph with no more than two cut-edges has a perfect matching, which shows that all fullerene graphs have perfect matchings [3]. A k-matching in a graph G is a set of (not adjacent) k edges of G.

In the following section, we investigate the number of k-matching in some (3,6)-fullerene graphs in terms of the number of hexagons.

2 Main results

In this section, we state some new results on the number of low order matching in (3,6)-fullerene graphs. First, let's state the two following lemmas about three, four and six-length cycles.

Lemma 2.1. *Let G is a (3,6)-fullerene graph.*

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*Speaker

subjclass[2010]: 05C10, 05C90, 05C92

keywords: Matching, Fullerene graphs, Perfect matching

- *If C is a 3-cycle in G, then C must be a facial cycle.*
- *If G is a non-trivial (3,6)-fullerene graph then it has no cycle of length four.*
- *If C is a 6-cycle in G, then C must be a facial cycle.*

Lemma 2.2. *For a trivial (3,6)-fullerene graph G, if C is a 4-cycle in G, then C is the either*

- *The boundary of dual triangles.*
- *The boundary of three triangles with a common vertex.*
- *The boundary of a dual triangle with hexagon layers.*

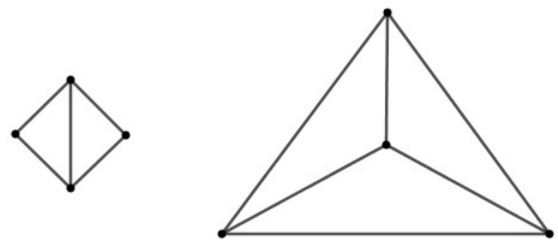

Figure 1: Dual triangle and three triangles with a common vertex.

Now, let G be a (3,6)-fullerene graph and t, h, n, and m be the number of triangles, hexagons, vertices, and edges of G, respectively. We denote the number of k-path (path of length k) by $P_k(G)$ and we have the following theorem.

Theorem 2.3. *Let G is a (3,6)-fullerene graph, then*

- $P_1(G) = 3h + 6$
- $P_2(G) = 6h + 12$
- $P_3(G) = 12h + 12$

Considering two trivial and non-trivial cases of G, we will have the following theorem.

Theorem 2.4. *Let G is a (3,6)-fullerene graph, then we have*

- $P_4(G) = \begin{cases} 22h + 8, & G \textit{ is trivial} \\ 24h + 24, & G \textit{ is non-trivial} \end{cases}$
- $P_5(G) = \begin{cases} 40h - 4, & G \textit{ is trivial} \\ 48h + 12, & G \textit{ is non-trivial} \end{cases}$

If $M(G,k)$ be the number of k-matching in G then we have the following theorem.

Theorem 2.5. *Let G is a (3,6)-fullerene graph, then we have*

$$M(G,1) = 3h + 6$$
$$M(G,2) = \frac{9}{2}h^2 + \frac{21}{2}h + 3$$
$$M(G,3) = \frac{9}{2}h^3 + \frac{9}{2}h^2 + 7h + 40$$

In the last Theorem of this paper, we get the values of $M(G,k)$, for $k = 4$.

Theorem 2.6. *Let G be a trivial (3,6)-fullerene graph, then*

$$M(G,4) = \begin{cases} \frac{27}{8}h^4 - \frac{27}{4}h^3 + \frac{33}{8}h^2 + \frac{61}{4}h - 102, & G \textit{ is trivial} \\ \frac{27}{8}h^4 - \frac{27}{4}h^3 + \frac{33}{8}h^2 + \frac{57}{4}h - 110, & G \textit{ is non-trivial} \end{cases}$$

In the case that $h = 2$, $M(G,4) = 5$.

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Lower Bounds for the Randić Index of Graph in Terms of Matching Number

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Abstract

Let G be a graph with the vertex set $\{v_1, \ldots, v_n\}$. The Randić index of G is defined as $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}$, where $d_u = d(u)$. In this talk, we prove that for every acyclic and unicyclic graph G with at least one edge, $R(G) > \frac{\alpha'(G)}{\sqrt{2}}$, for every subcubic G, $R(G) \geq \left(\frac{1}{\sqrt{3}} + \frac{1}{3}\right)\alpha'(G)$, and for every planar G, $R(G) > \frac{\alpha'(G)}{6\sqrt{42}}$, where $\alpha'(G)$ is the matching number of G.

Joint work with: S. Akbari, S. Ghasemi Nezhad, R. Ghazizadeh, J. Haslegraehve

1 Introduction

For a graph G, we denote the set of its vertices and edges by $V(G)$ and $E(G)$, respectively. The degree of a vertex v is denoted by d_u, and $N_G(v)$ denotes the set of all neighbors of v. For two arbitrary vertices u and v of this graph, by $u \sim v$, we mean that u and v are adjacent. We also denote the degree of a vertex u by d_u.

For a simple graph G, we define the Randić index by $R(G) = \sum_{i \sim j} \frac{1}{\sqrt{d_i d_j}}$. In 1975, the chemist Milan Randić proposed a topological index R under the name *branching index*, suitable for measuring the extent of branching of the carbon-atom skeleton of saturated hydrocarbons [1, 2]. Also, we denote the matching number of G by $\alpha'(G)$.

In this talk we prove that for every acyclic or unicyclic graph G with at least one edge, $R(G) > \frac{\alpha'(G)}{\sqrt{2}}$, for every subcubic graph, $R(G) \geq \left(\frac{1}{\sqrt{3}} + \frac{1}{3}\right)\alpha'(G)$, and for every planar G, $R(G) \geq \frac{\alpha'(G)}{6\sqrt{42}}$.

2 Main results

Let us start with the following result.

Theorem 2.1. *If T is a tree of order at least two, then $R(T) > \frac{\alpha'(T)}{\sqrt{2}}$.*

*Speaker

subjclass[2020]: 05C90, 05C92

keywords: Randić index, matching number, subcubic graphs

Then, we have two immediate corollaries.

Corollary 2.2. *If F is a forest with at least one edge, then $R(F) > \frac{\alpha'(F)}{\sqrt{2}}$.*

Corollary 2.3. *For a uni cyclic graph U, we have $R(U) > \frac{\alpha'(G)}{\sqrt{2}}$.*

The following interesting result was proved in [3].

Theorem 2.4. *Let G be a graph of order n. Then $R(G) \geq \frac{\sqrt{\delta\Delta}}{\delta+\Delta}n$.*

Now, we prove the following theorem.

Theorem 2.5. *Let G be a subcubic graph of order n. Then $R(G) \geq \left(\frac{1}{\sqrt{3}} + \frac{1}{3}\right)\alpha'(G)$ and equality holds if and only if $G = H \circ K_2$, for even n and $G = \left(H \circ K_2\right) \cup K_1$, for odd n, where H is a 2-regular graph.*

At the end, we prove the following theorem.

Theorem 2.6. *Let $c > 0$ be a real number and G be a graph. Let $V^+ = \{v \in V(G) | d_v > c\}$ and $V^- = \{v \in V(G) | d_v \leq c\}$. Then if the average degree of the vertices in $G[V^+]$ is at most c, we have $R(G) \geq \frac{\alpha'(G)}{c\sqrt{c(c+1)}}$.*

Corollary 2.7. *For a planar graph G, we have $R(G) \geq \frac{\alpha'(G)}{6\sqrt{42}}$.*

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The $\mathbb{N}_k$-valued Roman domination numbers of graphs

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Abstract

Let k be a positive integer and G be a simple graph with vertex set V. A $\mathbb{N}_k$*-valued Roman dominating function* (in abbreviation, $\mathbb{N}_k$-RDF) on G is a labeling $f : V \to \{0, 1, 2, \dots, k\}$ such that for every vertex $v \in V$ with label 0, there is a vertex $u \in V$ with label $f(u) \geq 1$ at distance at most $f(u) - 1$ from each other. The *weight* of a $\mathbb{N}_k$-valued Roman dominating function f is the value $\omega(f) = \sum_{v \in V} f(v)$ over all such functions f. The $\mathbb{N}_k$*-valued Roman domination number* of a graph G, denoted by $\Gamma_k(G)$, equals the minimum weight of a $\mathbb{N}_k$-valued Roman dominating function on G. Note that the $\mathbb{N}_2$-valued Roman domination number $\Gamma_2(G)$ is the usual Roman domination number $\gamma_R(G)$. In this paper, we investigate properties of the $\mathbb{N}_k$-valued Roman domination number.

1 Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Denote by K_n the *complete graph*, by C_n the *cycle* and by P_n the *path* of order n, respectively. The *complement* of a graph G is denoted by $\bar{G}$. Given two graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \varnothing$, the *disjoint union* is the graph $G_1 \cup G_2$ with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. For two vertices x and y, let $d(x, y)$ denote the distance between x and y in G. The eccentricity $e(v)$ of v is defined by $e(v) = \max\{d(v, x) \mid x \in V\}$. The radius $\mathrm{rad}(G)$ of G and the diameter $\mathrm{diam}(G)$ are defined as follows:

$$\mathrm{rad}(G) = \min\{e(v) \mid v \in V\} \quad \text{and} \quad \mathrm{diam}(G) = \max\{e(v) \mid v \in V\}.$$

The *girth* $g(G)$ of a graph G is the length of its shortest cycle.

Let k be a positive integer. For a vertex $v \in V$, the *open* k*-neighborhood* $N_{k,G}(v)$ is the set $\{u \in V \mid u \neq v \text{ and } d(u, v) \leq k\}$ and the *closed* k*-neighborhood* $N_{k,G}[v]$ is the set $N_{k,G}(v) \cup \{v\}$. The *open* k*-neighborhood* $N_{k,G}(S)$ of a set $S \subseteq V$ is the set $\bigcup_{v \in S} N_{k,G}(v)$, and the *closed neighborhood* $N_{k,G}[S]$ of S is the set $N_{k,G}(S) \cup S$. The k*-degree* of a vertex v is defined as $\deg_{k,G}(v) = |N_{k,G}(v)|$. The minimum and maximum k-degree of a graph G are denoted by $\delta_k(G)$ and $\Delta_k(G)$, respectively. If $\delta_k(G) = \Delta_k(G)$, then the graph G is called *distance-k-regular*.

*Speaker

subjclass[2010]: 05C31, 05C60

keywords: Roman Domination, $\mathbb{N}_k$-valued Roman dominating function, $\mathbb{N}_k$-valued Roman domination number

2 Main results

Lemma 2.1. [2] For any tree T of order $n \geq 3$, $\gamma_R(T) \geq 4n/5$.

Lemma 2.2. [5] For any connected graph G with diameter d and any positive integer k,

$$\Gamma_k(G) \geq \left\lceil \frac{1}{2}\left(d+1+\left\lceil \frac{d+1}{2k-1}\right\rceil\right)\right\rceil.$$

Lemma 2.3. [1] Let G be a graph of order n. Then $\gamma_R(G) = n$ if and only if $G = rK_1 \cup sK_2$ for some integers $r, s \geq 0$.

The next observations are straightforward to verify.

Proposition 2.4. For any graph G and any positive integer $k \geq 2$,

$$\Gamma_k(G) \leq \ldots \leq \Gamma_3(G) \leq \Gamma_2(G) = \gamma_R(G).$$

Proposition 2.5. For any graph G of order n and maximum degree $\Delta \geq 1$,

$$\Gamma_k(G) \geq \left\lceil \frac{kn}{(k-1)\Delta+1}\right\rceil.$$

Proof. Since each vertex with value k dominates at most $\Delta(k-1)+1$ vertices, so the desired result obtain. □

The special case $k = 2$ of Proposition 2.5 can be found in [3].

Proposition 2.6. Let $k \geq 2$ be an integer and let G be a graph of order n. Then $\Gamma_k(G) = n$ if and only if $G = rK_1 \cup sK_2$ for some integers $r, s \geq 0$.

Proof. If $G = rK_1 \cup sK_2$ for some integers $r, s \geq 0$, then obviously $\Gamma_k(G) = n$. Conversely, assume that $\Gamma_k(G) = n$. Then Observation 2.4 leads $\gamma_R(G) = n$. Thus from Proposition 2.3, $G = rK_1 \cup sK_2$ for some integers $r, s \geq 0$. □

Proposition 2.7. If $k \geq 2$ is an integer and G is a graph of order n with $\Delta_{k-1}(G) \geq 1$, then $\Gamma_k(G) \leq n - \Delta_{k-1}(G) + k - 1$.

Proof. Let v be a vertex of G such that $\deg_{k-1,G}(v) = \Delta_{k-1}(G)$.
Then $f = (N_{k-1,G}(v), V(G) - N_{k-1,G}[v], \varnothing, \ldots, \varnothing, v)$ is a $\mathbb{N}_k$-RDF on G with weight $n - \Delta_{k-1}(G) + k - 1$ and therefore $\Gamma_k(G) \leq n - \Delta_{k-1}(G) + k - 1$. □

Proposition 2.8. Let G be a graph of order $n \geq 4$. Then $\Gamma_3(G) = 3$ if and only if either $\mathrm{diam}(G) = 3$ or $\mathrm{diam}(G) = 4$.

Proof. If $\mathrm{diam}(G) = 3$ or $\mathrm{diam}(G) = 4$, then by Observation **??**, $\Gamma_3(G) \leq 3$. Moreover, it follows from Proposition 2.2 that $\Gamma_3(G) \geq 3$. Thus $\Gamma_3(G) = 3$.

Now suppose that $\Gamma_3(G) = 3$. If $\mathrm{diam}(G) \leq 2$, then it is not hard to see that $\Gamma_3(G) = 2$. Let $\mathrm{diam}(G) \geq 3$. If $\mathrm{diam}(G) \geq 5$, then by Proposition 2.2, we have $\Gamma_3(G) \geq 4$, a contradiction. □

The following two classes of graphs achieve the lower bound of Proposition 2.5. Proof is straightforward and is omitted.

Proposition 2.9. For $n \geq 3$,

$$\Gamma_k(C_n) = \Gamma_k(P_n) = \left\lceil \frac{kn}{2k-1}\right\rceil.$$

For the class of complete multipartite graphs $K_{m_1,\ldots,m_n}$ there are two cases to consider.

Proof. Let $C_n = (v_1, v_2, \ldots, v_n)$ be a cycle on n vertices. By Proposition 2.5, $\Gamma_k(C_n) \geq$

Case 1. $n \equiv 0 \pmod{2k-1}$.

Define $g : V(C_n) \to \{0, 1, 2, \ldots, k\}$ by $g(v_i) = k$ if $i \equiv k (\mathrm{mod}\ 2k-1)$ and $g(v_i) = 0$ otherwise. It is easy to see that g is a $\mathbb{N}_k$-RDF on C_n with weight $\frac{kn}{2k-1}$ and hence $\Gamma_k(C_n) = \frac{kn}{2k-1}$.

Case 2. $n \equiv t \neq 0 (\mathrm{mod}\ 2k-1)$.

Define $g : V(C_n) \to \{0, 1, 2, \ldots, k\}$, by $g(v_i) = k$ for $1 \leq i \leq \frac{n-t}{2k-1}$, $g(v_{\frac{n-t}{2k-1} + \lfloor \frac{t}{2} \rfloor}) = \lfloor \frac{t}{2} \rfloor + 1$ and $g(v_i) = 0$ otherwise. Then, g is a $\mathbb{N}_k$-RDF on C_n with weight $\frac{k(n-t)}{2k-1} + \lfloor By Proposition 2.5, \Gamma_k(C_n) \geq \frac{kn}{2k-1} = \frac{k(n-t)}{2k-1} + \frac{kt}{2k-1} = \frac{k(n-t)}{2k-1} + \frac{t}{2} + \frac{t/2}{2k-1}. Since \Gamma_k(C_n)$ is an integer, we have $\Gamma_k(C_n) \geq \frac{k(n-t)}{2k-1} + \lfloor$ □

Proposition 2.10. *For $n \geq 2$,*

$$\Gamma_k(P_n) = \begin{cases} \frac{kn}{2k-1} & n \equiv 0 \pmod{2k-1} \\ \frac{k(n-t)}{2k-1} + \lfloor \frac{t}{2} \rfloor + 1 & n \equiv t \neq 0 \pmod{2k-1} \end{cases}$$

Proposition 2.11. If $k \geq 2$ is an integer and G a connected graph of order $n \geq 3$ and $\infty > g(G) \geq 2k-1$, then

$$\Gamma_k(G) \geq \left\lceil \frac{kg(G)}{2k-1} \right\rceil.$$

Proof. If G is an n cycle then the result follows from Proposition 2.9. Now, let C be a cycle of length $g(G)$ in G and let $f = (V_0, V_1, V_2, \ldots, V_k)$ be a $\mathbb{N}_k$-RDF. Since each vertex in $V(C)$ dominates at most $2k-1$ vertex of $V(C)$, we have

$$\Gamma_k(G) = \sum_{i=1}^{k} i|V_i| \geq k|V_k| \geq \frac{kg(G)}{2k-1}.$$

This leads to the desired bound, and the proof is complete. □

The special case $k = 2$ of Proposition 2.11 can be found in [5].

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Connected edge cover polynomial of a graph

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Abstract

Let $G = (V, E)$ be a simple graph of order n and size m. A connected edge cover set of a graph is a subset S of edges such that every vertex of the graph is incident to at least one edge of S and the subgraph induced by S is connected. The connected edge cover polynomial of G is the polynomial $E_c(G, x) = \sum_{i=1}^{m} e_c(G, i)x^i$, where $e_c(G, i)$ is the number of connected edge cover set of size i. We investigate this polynomial for some graphs.

1 Introduction

Let $G = (V, E)$ be a simple graph. The *order* and the size of G is the number of vertices and the number of edges of G, respectively. For every graph G with no isolated vertex, an edge covering of G is a set of edges of G such that every vertex is incident with at least one edge of the set. In other words, an edge covering of a graph is a set of edges which together meet all vertices of the graph. A minimum edge covering is an edge covering of the smallest possible size. The edge covering number of G is the size of a minimum edge covering of G and denoted by $\rho(G)$. We let $\rho(G) = 0$, if G has some isolated vertices. For a detailed treatment of these parameters, the reader is referred to [1, 2, 3, 4]. Let $\mathscr{E}(G, i)$ be the family of all edge coverings of a graph G with cardinality i and let $e(G, i) = |\mathscr{E}(G, i)|$. The *edge cover polynomial* $E(G, x)$ of G is defined as

$$E(G, x) = \sum_{i=\rho(G)}^{m} e(G, i)x^i,$$

where $\rho(G)$ is the edge covering number of G. Also, for a graph G with some isolated vertices we define $E(G, x) = 0$. Let $E(G, x) = 1$, when both order and size of G are zero (see [1]).

In [1] authors have characterized all graphs whose edge cover polynomials have exactly one or two distinct roots and moreover they proved that these roots are contained in the set $\{-3, -2, -1, 0\}$. In [2], authors constructed some infinite families of graphs whose edge cover polynomials have only roots -1 and 0. Also, they studied the edge coverings and edge cover polynomials of cubic graphs of order 10. As a consequence , they have shown that the all cubic graphs of order 10 (especially the Petersen graph) are determined uniquely by their edge cover polynomials.

Motivated by the edge cover number and polynomial, we consider the following definition.

[*]Speaker

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Definition 1.1. *A connected edge cover set of graph G is a subset S of edges such that every vertex of G is incident to at least one edge of S and the subgraph induced by S is connected. The connected edge cover number of G is equal to the minimum cardinality of the connected edge cover and is denoted by $\rho_c(G)$.*

Also, we state the following definition for the connected edge cover polynomial.

Definition 1.2. *The connected edge cover polynomial of G is the polynomial*

$$E_c(G,x) = \sum_{i=1}^{m} e_c(G,i)x^i,$$

where $e_c(G,i)$ is the number of connected edge cover set of size i.

For two graphs G and H, the corona $G \circ H$ is the graph arising from the disjoint union of G with $|V(G)|$ copies of H, by adding edges between the ith vertex of G and all vertices of ith copy of H. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex u and a pendant edge $\{v,u\}$ are added. It is easy to see that the corona operation of two graphs does not have the commutative property.

In this paper, we obtain the connected edge cover polynomial for certain graphs.

2 Main results

Here, we state some new results on the connected edge cover number and the edge cover polynomial.

Theorem 2.1. *(i) For every natural numbers n, $\rho_c(K_n) = n-1$.*
(ii) For every natural numbers $n \geq 4$, $E_c(K_n,x) = E(K_n,x) - \sum_{i=\lceil n/2 \rceil}^{n-2} e(K_n,i)x^i$.

Theorem 2.2. *For every natural numbers $n \geq 3$,*
(i) $\rho_c(C_n) = n-1$.
(ii) $E_c(C_n,x) = \sum_{i=n-1}^{n} \binom{n}{i} x^i$.
(iii) $\rho_c(P_n) = n-1$.
(iv) $E_c(P_n,x) = x^{n-1}$.

Theorem 2.3. *For every natural numbers $n \geq 2$,*
(i) $\rho_c(P_n \circ K_1) = 2n-1$.
(ii) $E_c(P_n \circ K_1,x) = x^{2n-1}$.
(iii) For every natural numbers $n \geq 3$, $\rho_c(C_n \circ K_1) = 2n-1$.
(ii) $E_c(C_n \circ K_1,x) = x^{2n} + 2nx^{2n-1}$.

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On the distance transitivity of the bipartite Kneser graphs

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Abstract

In this paper, we study a family of graphs which is related to Johnson graphs. These graphs are called bipartite Kneser graphs. Let n and k be integers with $n > k \geq 1$. We denote by $H(n,k)$ the bipartite Kneser graph, that is, a graph with the family of k-subsets and $(n-k)$-subsets of the set $[n] = \{1,2,...,n\}$ as its vertex-set, in which any two vertices are adjacent if and only if one of them is a subset of the other. Mirafzal (Mirafzal SM, The automorphism group of the bipartite Kneser graph, Proc. Indian Acad. Sci. (Math. Sci). Forthcoming Articles), proved that the automorphism group of the bipartite Kneser graph $H(n,k)$ is isomorphic to $\mathrm{Sym}([n]) \times \mathbb{Z}_2$. In this paper, we study the distance-transitivity and the diameter of the bipartite Kneser graphs. In fact, we will show that for which n, the bipartite Kneser graph $H(n,k)$ is a distance-transitive graph.

1 Introduction

In this paper, a graph $\Gamma = (V,E)$ is considered as a finite undirected simple graph, where $V = V(\Gamma)$ is the vertex-set and $E = E(\Gamma)$ is the edge-set. The degree of each vertex $v \in V(\Gamma)$, is the number of neighbours of v in Γ and denoted by $\deg(v)$. A graph $\Gamma = (V,E)$ is called k-regular (or regular graph with degree k), if $\deg(v) = k$ for every $v \in V(\Gamma)$. Let u,v be two vertices in (connected) graph Γ. Then the length of the shortest path from u to v is called the distance between u,v and denoted by $d_\Gamma(u,v)$ (if there is no misunderstand, we use $d(u,v)$ instead of $d_\Gamma(u,v)$). For all the terminologies and notations which are not defined here, we follow [3, 5, 9].

Definition 1.1. *Let $[n] = \{1,\cdots,n\}$ be a set of size n. Let m be an integer such that $2m \leq n$. Then the Johnson graph $J(n,m)$ is the graph with the vertex-set consist of all m-subsets (subsets of size m) of $[n]$, and two vertices u,v are adjacent if and only if $|u \cap v| = m-1$.*

The number of vertices of $J(n,m)$ is equal to $\binom{n}{m}$. Furthermore, the Johnson graph $J(n,m)$ is a regular graph with degree $m(n-m)$ (see [3] or [5]). For more information about Johnson graphs you can see [1, 7].
The Kneser graph $K(n,k)$ is the graph with the vertex-set consist of all k-subsets of $[n]$. Two vertices are adjacent if they are disjoint when considered as k-sets.

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Definition 1.2. *For a positive integer $n > 1$, let $[n] = \{1, 2, ..., n\}$ and V be the set of all k-subsets and $(n-k)$-subsets of $[n]$. The bipartite Kneser graph $H(n,k)$ has V as its vertex-set, and two vertices A, B are adjacent if and only if $A \subset B$ or $B \subset A$. That is, A is adjacent to B if A and B^c are disjoint, where B^c is defined to be $B^c = [n] - B$.*

The bipartite Kneser graph $H(n,k)$ is a regular bipartite graph with the degree $\binom{n-k}{k}$. For more information about bipartite Kneser graphs, you can see [9, 10].

Ya-Chen Chen [4] showed that the Kneser graph $K(n,k)$ is Hamiltonian for $n \geq 3$, where $\binom{3k}{k}$ is odd. Also in [4] it is stated that, Savage and Shields showed that $H(2k+1,k)$ is Hamiltonian for $k \leq 15$. The bipartite Kneser graph $H(2n-1,n-1)$ is known as the middle cube MQ_{2n-1} [6] or regular hyper-star graph $HS(2n,n)$ [8].

Let $\Gamma = (V,E)$ be a graph. Then the mapping $f : V \longrightarrow V$ is called an automorphism of Γ if and only if f is a bijection map and f preserve the adjacency of vertices in Γ. The set of all automorphisms of Γ with the operation of composition of functions is a group, called the automorphism group of Γ and denoted by $\mathrm{Aut}(\Gamma)$. Oftentimes, determining the automorphism group of the graphs is difficult. There are various papers in the literature, and some of the recent works appear in the references [7, 9].

The graph Γ is called vertex-transitive, if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. For $v \in V(\Gamma)$ and $G = \mathrm{Aut}(\Gamma)$, the stabilizer subgroup G_v is the subgroup of G consisting of all automorphisms that fix v. We say that Γ is symmetric (or arc-transitive) if, for all vertices u, v, x, y of Γ such that u and v are adjacent, also, x and y are adjacent, there is an automorphism α in $\mathrm{Aut}(\Gamma)$ such that $\alpha(u) = x$ and $\alpha(v) = y$. We say that Γ is distance-transitive if, for all vertices u, v, x, y of Γ such that $d(u,v) = d(x,y)$, there is an automorphism $\beta \in \mathrm{Aut}(\Gamma)$ satisfying $\beta(u) = x$ and $\beta(v) = y$.

In this paper, we study about the distance-transitivity of bipartite Kneser graphs. In [3] we can see that the Kneser graphs and Johnson graphs are distance-transitive graph.

2 Main results

In this section, we study about the diameter and distance-transitivity of the bipartite Kneser graphs. By Lemma 3.2. from [9], the bipartite Kneser graph $H(n,k)$ is a symmetric graph. Furthermore, by theorem 3.9. of [9] we know that the automorphism group of bipartite Kneser graph $H(n,k)$ is isomorphic to $\mathrm{Sym}([n]) \times \mathbb{Z}_2$.

Let $\Gamma = (V,E)$ be a graph and $v \in V(\Gamma) = V$ be a vertex of Γ. Let $\mathrm{diam}(\Gamma) = d$, then for each $i = 0, \cdots, d$ we denote the set of all vertices at distance i from v, by $\Gamma_i(v)$. In other words,

$$\Gamma_i(v) = \{u \in V \mid d(u,v) = i\}.$$

Lemma 2.1. *[2] A connected graph Γ with the diameter d and automorphism group $G = \mathrm{Aut}(\Gamma)$ is distance-transitive if and only if it is a vertex-transitive graph and the vertex-stabilizer G_v acts transitively on the set $\Gamma_i(v)$, for all $i \in \{0, \cdots, d\}$, and each $v \in V(\Gamma)$.*

Proposition 2.2. *Let k be an integer and $n = 2k+1$. Let $\Gamma = (V,E) = H(n,k)$ be bipartite Kneser graph with vertex-set partition $V = V_1 \cup V_2, V_1 \cap V_2 = \varnothing$, where $V_1 = \{v \subset [n] \mid |v| = k\}$ and $V_2 = \{v \subset [n] \mid |v| = k+1\}$. Let $u, v \in V$ be two vertices in V such that $|u \cap v| = i$, then we have the following cases;*

(1) If $u, v \in V_1$, then $d(u,v) = 2(k-i)$;

(2) If $u \in V_1$ and $v \in V_2$, then $d(u,v) = 2(k-i)+1$;

(3) If $u, v \in V_2$, then $d(u,v) = 2(k+1-i)$.

Theorem 2.3. *Let $n = 2k+1$.Then the graph $H(n,k)$ is a distance-transitive graph.*

Proposition 2.4. *Let $k \geq 2$ be an integer and $n \geq 3k$. Then the diameter of bipartite Kneser graph $H(n,k)$ is equal to 4.*

Proposition 2.5. *Let $k \neq 1$ be an integer and $2k+1 < n < 3k$. Then the diameter of bipartite Kneser graph $H(n,k)$ is equal to 6.*

Proposition 2.6. *Let $n \geq 3$ be an integer. Then the diameter of bipartite Kneser graph $H(n,1)$ is equal to* 3.

Corollary 2.7. *Let $k \geq 2$ be an integer and $n \neq 2k+1$. Then the bipartite Kneser graph $H(n,k)$ is not distance-transitive.*

Theorem 2.8. *Let $n \geq 3$ be an integer. Then the bipartite Kneser graph $\Gamma = H(n,1)$ is a distance-transitive graph.*

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